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EFFECT OF PAIRING CORRELATIONS ON NUCLEAR PROPERTIES

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Synopsis.

A method analogous to that developed in the new theory of superconductivity is applied to nuclei in order to investigate the influence of the coherent pairing interaction on various nuclear properties, especially on collective motion. The finite size effects, in particular the shell structure of the single-particle levels, are considered. The pairing correlation between two nucleons in states of opposite angular momentum projections is taken into account by means of a canonical transformation from the original interacting nucleons to new independent quasi-particles.

For strongly deformed nuclei, the moment of inertia is rather sensitive to the effect of pairing correlations and is found to be reduced from the value for rigid rotation by a factor of the order of that observed. For nuclei in regions near closed shells, the pairing correlations give rise to a spherical equilibrium shape and low energy vibrational modes of excitations. The vibrational frequencies and inertial parameters obtained from the present model are in qualitative agreement with experimental data and fit the observed trends.

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Introduction

The Fermi gas model, which neglects the interaction between nucleons, is the simplest microscopic model of the nucleus. The development of nuclear models has progressed by taking into account certain parts of the nucleon-nucleon interaction. The great successes of the shell model, in which the nucleons are assumed to move independently in a certain average potential, showed that evidently the main part of interaction can be treated as a spherically symmetrical, self-consistent field. In the unified nuclear model, developed by A. Bohr, B. Mottelson,[1, 2] and others[3], it is assumed that, from the remaining part of the nucleon-nucleon interaction, an additional self-consistent part may be extracted, which is non-spherical and time-dependent. This procedure makes it possible to explain many of the regularities in the low-lying nuclear levels in the language of collective excitations.

However, the real interaction between nucleons cannot be reduced simply to a self-consistent field. After separation of the self-consistent part, there remains some interaction between the particles. This residual interaction is rather weak, but it may play an important role in various nuclear properties[4, 5].

Recent work in the theory of superconductivity[6–8] has shown that even small interactions between Fermi particles may give rise to a basic change in the properties of the system, provided this interaction has a correlated coherent character. In a superconductor, the correlations between electrons arise from the interaction with the lattice vibrations and make possible quasi-bound states of electron pairs with equal and opposite momenta near the Fermi surface[9]. This leads to a modification of the Fermi sea and to the appearance of a gap in the originally continuous energy spectrum of the system.

After the appearance of the new theory of superconductivity the suggestion was made[10] that the energy gap found in the spectra of even-even nuclei is caused by correlation effects of a similar type to those considered for the electron system in superconductors. Such correlations may also affect other nuclear properties, which have no analogue in superconductors, connected with the finite size of nuclei and the shell structure of the single particle levels. It is the aim of the present paper to investigate the effect of the pairing correlation on various nuclear phenomena, in particular, on collective nuclear excitations.

We extend the method of the new theory of superconductivity developed by N. Bogolyubov[7] in order to apply it to the nuclear system. The physical basis of the analogy is the similarity between the pairing energy of two nucleons with opposite projections of angular momentum and quasi-bound states of electron pairs with equal and opposite momenta. The correlation effect between nucleons is taken into account by means of a canonical transformation from the original interacting nucleons to new independent quasi-particles — the elementary excitations. The ground state of the system in terms of the new quasi-particles is the "vacuum" state. The essential part of the pairing correlation enters into the "vacuum" energy and into the intrinsic structure of the quasi-particles. Therefore, even if the residual interaction between the quasi-particles is neglected, one may investigate the influence of the correlation interaction on various nuclear properties. The general idea of the treatment is to take into account the coherent part of the residual internucleon interaction, but, at the same time, to retain the simple description afforded by the independent-particle model (with a type of quasi-particles).

In the first part, we consider the general formulation of the problem and select the canonical transformation required to take into account the effects of correlation between nucleons.

An explicit solution of the equation for the transformation coefficients is given in the second part. Here are also given, in the approximation of independent quasi-particles, the energy and the wave function of the ground state of the system and of the single-particle excited states.

The problems concerning the nuclear equilibrium shape and collective excitations are considered in the third part. Here, the moment of inertia for nuclear rotations and the inertial parameter and restoring force for the quadrupole vibrations of spherical nuclei are found within the framework of the cranking model.

I. Canonical Transformation

1. Hamiltonian

We consider a system of nucleons which are moving in a certain axially symmetric self-consistent well. (For simplicity, we do not distinguish between neutrons and protons). As basic functions of the second quantization representation we choose the wave functions of a nucleon in this well. States, which differ only in the sign of the projections of angular momentum along the symmetry axis, are degenerate. We call such states "conjugate" states and mark them with the index $k\sigma = (k+;\ k-)$*.

The wave functions of the conjugate states are assumed to transform into each other by complex conjugation and exchange of the spinor components**.

Let us introduce the Fermi operators $a^{+}_{k\sigma};\ a_{k\sigma}$ which create and destroy a particle in the state $k\sigma$. The Hamiltonian for the system of interacting particles is then

$$\left.\begin{aligned} H' &= \sum_{k} \varepsilon_k \left(a^{+}_{k+} a_{k+} + a^{+}_{k-} a_{k-}\right) \\ &-\frac{1}{2}\sum_{(k,\sigma)} \langle k_1\sigma_1 k_2\sigma_2 \,|\, G \,|\, k_2'\sigma_2' k_1'\sigma_1' \rangle\, a^{+}_{k_1\sigma_1} a^{+}_{k_2\sigma_2} a_{k_2'\sigma_2'} a_{k_1'\sigma_1'}, \end{aligned}\right\} \quad (1)$$

where ε_k is the single-particle energy in the k-th state. (The sign of G is chosen to be positive for an attractive interaction).

The Hamiltonian (1) describes a system with a fixed number of particles N. Therefore, in a perturbation treatment in which H' is split into two parts, each of these parts must commute with N. The problem is essentially simplified if we make a transition from the system with fixed N ("N-system")

* In fact, even symmetry of reflection in a plane is enough for the definition of the conjugate states. We speak of axial symmetry only for definiteness.

** If $\psi_+ = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, then $\psi_- = \begin{pmatrix}\psi_2^*\\ -\psi_1^*\end{pmatrix}$. The transformation $\psi_+ \to \psi_-$ is equivalent to the time reversal T.

to one with a fixed value of the chemical potential λ ("λ-system"), which is described by the Hamiltonian

$$H = H' - \lambda N. \tag{2}$$

The choice of λ determines only the average value of N in the λ-system. Therefore, the solution which corresponds to the Hamiltonian, (2), will describe only average properties of nuclei and does not pretend to describe the individual nuclear properties for which one needs a fixed value of N. As will be shown later, the uncertainty in the value of N is small. In practice, the averaging is done only over a few neighbouring nuclei, either all even or all odd.

2. Canonical Transformation

Following the analogy with the model of a superconductor, we choose a canonical transformation of the form given by N. Bogolyubov[7]. In our case, however, it is necessary to consider a transformation of a more general type, because the interparticle interaction in (1) in general contains not only pairing interactions, but a certain supplementary self-consistent field. Therefore, we perform the following preliminary transformation to remove the self-consistent field:

$$b_{\nu+} = \sum_k \varphi^*_{k\nu} a_{k+}; \quad b_{\nu-} = \sum_k \varphi_{k\nu} a_{k-}, \tag{3}$$

where the coefficients satisfy the conditions

$$\sum_k \varphi^*_{k\nu} \varphi_{k\nu'} = \delta_{\nu\nu'}; \quad \sum_\nu \varphi^*_{k\nu} \varphi_{k'\nu} = \delta_{kk'}. \tag{4}$$

The conjugate relationship defined above is preserved by this transformation. Inverting (3), we obtain

$$a_{k+} = \sum_\nu \varphi_{k\nu} b_{\nu+}; \quad a_{k-} = \sum_\nu \varphi^*_{k\nu} b_{\nu-}. \tag{5}$$

After the transformation to the new operators the Hamiltonian (2) takes the form

$$\left.\begin{aligned} H = &\sum_{\nu\nu'} (\varepsilon_{\nu\nu'} - \lambda\delta_{\nu\nu'}) (b^+_{\nu+} b_{\nu'+} + b^+_{\nu'-} b_{\nu-}) \\ &- \frac{1}{2} \sum_{\nu\sigma} \langle \nu_1 \sigma_1 \nu_2 \sigma_2 | G | \nu'_2 \sigma'_2 \nu'_1 \sigma'_1 \rangle b^+_{\nu_1\sigma_1} b^+_{\nu_2\sigma_2} b_{\nu'_2\sigma'_2} b_{\nu'_1\sigma'_1}, \end{aligned}\right\} \tag{6}$$

where

$$\varepsilon_{\nu\nu'} = \sum_k \varepsilon_k \varphi_{k\nu}^* \varphi_{k\nu'}, \tag{7}$$

and the interaction matrix element is taken between the new states.

The self-consistent field is caused by the correlation between a great number of states. The character of the transformation (3) has this physical interpretation. After separation of the self-consistent field, each state is assumed to be correlated only with its conjugate state. The interaction mixes the states of the conjugate pair. In order to take into account this effect we introduce, instead of $b_{\nu\sigma}$, the new Fermi operators

$$\left.\begin{aligned} \alpha_\nu &= U_\nu b_{\nu+} - V_\nu b_{\nu-}^+, \\ \beta_\nu &= U_\nu b_{\nu-} + V_\nu b_{\nu+}^+, \end{aligned}\right\} \tag{8}$$

where U_ν and V_ν are real numbers which obey the condition

$$U_\nu^2 + V_\nu^2 = 1. \tag{9}$$

The special choice of $U_\nu = 1$; $V_\nu = 0$ for the states above the Fermi surface ($\varepsilon_\nu > \lambda$), and $U_\nu = 0$; $V_\nu = 1$ for $\varepsilon_\nu < \lambda$, leads to the well-known transformation from particles and holes to elementary excitations. The completely occupied Fermi sea goes to the "vacuum". In general, the new particles (α; β) are a superposition of a particle and a hole, and the "vacuum" corresponds to a modification of the Fermi sea.

The transformation inverse to (8) has the form

$$\left.\begin{aligned} b_{\nu+} &= U_\nu \alpha_\nu + V_\nu \beta_\nu^+, \\ b_{\nu-} &= U_\nu \beta_\nu - V_\nu \alpha_\nu^+. \end{aligned}\right\} \tag{10}$$

Inserting (10) into (6), we obtain a Hamiltonian with the following structure:

$$H = U + H_{20} + H_{11} + H_{\text{int}}. \tag{11}$$

Here, U is a constant term

$$U = \sum_\nu \left(\varepsilon_{\nu\nu} - \lambda - \frac{1}{2} \sum_{\nu_1} \langle \nu\nu_1 | \overline{G} | \nu_1 \nu \rangle V_{\nu_1}^2 \right) 2 V_\nu^2 - \sum_{\nu\nu_1} \langle \nu\nu | G | \nu_1 \nu_1 \rangle U_{\nu_1} V_{\nu_1} U_\nu V_\nu. \tag{12}$$

The terms H_{20} and H_{11} are quadratic in the new operators

$$\left.\begin{aligned} H_{20} = & \sum_{\nu\nu'} \Big\{ \Big(\varepsilon_{\nu\nu'} - \lambda \delta_{\nu\nu'} - \sum_{\nu'} \langle \nu\nu_1 | \overline{G} | \nu_1 \nu' \rangle V_{\nu_1}^2 \Big) (U_\nu V_{\nu'} + V_\nu U_{\nu'}) \\ & - \sum_{\nu_1} \langle \nu\nu' | G | \nu_1 \nu_1 \rangle U_{\nu_1} V_{\nu_1} (U_\nu U_{\nu'} - V_\nu V_{\nu'}) \Big\} (\alpha_\nu^+ \beta_{\nu'}^+ + \beta_\nu \alpha_{\nu'}), \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} H_{11} = \sum_{\nu\nu'} \Big\{ \Big(\varepsilon_{\nu\nu'} - \lambda\delta_{\nu\nu'} - \sum_{\nu'} \langle \nu\nu_1 | \overline{G} | \nu_1 \nu' \rangle V_{\nu_1}^2 \Big) (U_\nu U_{\nu'} - V_\nu V_{\nu'}) \\ + \sum_{\nu_1} \langle \nu\nu' | G | \nu_1 \nu_1 \rangle U_{\nu_1} V_{\nu_1} (U_\nu V_{\nu'} + V_\nu U_{\nu'}) \Big\} (\alpha_\nu^+ \alpha_{\nu'} + \beta_{\nu'}^+ \beta_\nu). \end{aligned}\right\} \quad (14)$$

The matrix elements in (12)—(14) have the form

$$\left.\begin{aligned} &\langle \nu_1 \nu_2 | G | \nu_2' \nu_1' \rangle = \langle \nu_1 + \nu_2 - | G | \nu_2' - \nu_1' + \rangle - \langle \nu_1 + \nu_2 - | G | \nu_1' + \nu_2' - \rangle \\ &\langle \nu_1 \nu_2 | \overline{G} | \nu_2' \nu_1' \rangle = \langle \nu_1 \nu_2 | G | \nu_2' \nu_1' \rangle \\ &\qquad + \langle \nu_1 + \nu_2' + | G | \nu_2 + \nu_1' + \rangle - \langle \nu_1 + \nu_2' + | G | \nu_1' + \nu_2 + \rangle. \end{aligned}\right\} \quad (15)$$

The last term in (11), H_{int}, contains products of four operators and describes the interaction between the new particles (α; β). It may be written in the form

$$H_{\text{int}} = H_{40} + H_{31} + H_{22}, \quad (16)$$

where the subscripts indicate the relative numbers of creation and destruction operators in the corresponding term, e. g., the term H_{40} describes the creation of four particles from the vacuum (or the inverse process) and so on. (The explicit expression of H_{int} is given in Appendix A). In the following, we consider mainly the independent quasi-particle model, neglecting the interaction term H_{int}. Effects of this term will be briefly discussed at the end of Part II.

3. Choice of the Transformation Coefficients

Neglecting the interaction between the new particles, let us consider the Hamiltonian

$$H_0 = U + H_{20} + H_{11}. \quad (17)$$

Following the programme outlined in the Introduction, we choose the coefficients of the canonical transformations so as to make H_0 correspond to an independent-particle system. This is possible only if $H_{20} = 0$ and H_{11} is a function only of the occupation numbers of the new particles $\alpha_\nu^+ \alpha_\nu$ and $\beta_\nu^+ \beta_\nu$.* From (13) it follows that the first condition leads to the equation

* The condition $H_{20} = 0$ may be easily shown to be exactly equivalent to the requirement of a minimum "vacuum" energy U. Therefore, the ground state of the system in terms of the new particles is a "vacuum" state. The excited states are characterized by definite numbers of new particles, elementary excitations.

$$\left.\begin{aligned} h_{\nu\nu'} \equiv \Big(\varepsilon_{\nu\nu'} - \lambda\delta_{\nu\nu'} - \sum_{\nu_1} \langle \nu\nu_1 | \bar{G} | \nu_1 \nu' \rangle V_{\nu_1}^2\Big) (U_\nu V_{\nu'} + V_\nu U_{\nu'}) \\ - \sum_{\nu_1} \langle \nu\nu' | G | \nu_1 \nu_1 \rangle U_{\nu_1} V_{\nu_1} (U_\nu U_{\nu'} - V_\nu V_{\nu'}) = 0. \end{aligned}\right\} \quad (18)$$

The solution of (18) for $\nu' \neq \nu$ is equivalent to the diagonalization of the matrix $h_{\nu\nu'}$. It can be carried out by arbitrary U_ν, V_ν only with an appropriate choice of the coefficients in the transformation (3), i. e., the states ν. The quantities U_ν, V_ν might then be determined from the diagonal part of (18). In the general case, the choice of the states ν depends on U_ν, V_ν, and the two transformations are not independent of each other. The quantity $h_{\nu\nu'}$ in (18) is a linear combination of two non-diagonal matrices

$$\tilde{\varepsilon}_{\nu\nu'} = \varepsilon_{\nu\nu'} - \sum_{\nu_1} \langle \nu\nu_1 | \bar{G} | \nu_1 \nu' \rangle V_{\nu_1}^2, \tag{19}$$

$$\Delta_{\nu\nu'} = \sum_{\nu_1} \langle \nu\nu' | G | \nu_1 \nu_1 \rangle U_{\nu_1} V_{\nu_1}. \tag{20}$$

From (19) it is seen that $\tilde{\varepsilon}_{\nu\nu'}$ is the energy of a particle in a self-consistent field*. The diagonalization of $\tilde{\varepsilon}_{\nu\nu'}$ corresponds to the transition to the single-particle eigenstates in this new field. Generally, it does not lead to the diagonalization of $\Delta_{\nu\nu'}$ (and, therefore, $h_{\nu\nu'}$). But, in many cases of practical interest, the diagonalization of the single-particle energy $\tilde{\varepsilon}_{\nu\nu'}$ gives rise to the following selection rule for the interaction matrix element:

$$\langle \nu\nu' | G | \nu_1 \nu_1 \rangle = 0 \quad \text{for} \quad \nu' \neq \nu. \tag{21}$$

(In the jj-shell model, $\nu \to m_j$ and Eq. (21) is a consequence of the conservation of the angular momentum). This makes both $\tilde{\varepsilon}_{\nu\nu'}$ and $\Delta_{\nu\nu'}$ simultaneously diagonal. There remain then in the sum in (14) only the terms with $\nu' = \nu$, and H_{11} takes the form of a Hamiltonian for independent particles. For simplicity, we restrict ourselves to this case**.

Assuming the diagonalization of $\tilde{\varepsilon}_{\nu\nu'}$ and $\Delta_{\nu\nu'}$ to be fulfilled, we obtain from (12)—(14)

$$U = \sum_\nu (\tilde{\varepsilon}_\nu - \lambda)\, 2\, V_\nu^2 - \sum_\nu \Delta_\nu U_\nu V_\nu + \sum_{\nu\nu'} \langle \nu\nu' | \bar{G} | \nu' \nu \rangle V_{\nu'}^2 V_\nu^2, \tag{22}$$

* If the Fermi sea is not modified, the sum in (19) spreads only over the occupied states for which $V_\nu^2 = 1$. In the general case, V_ν^2 describes the average distribution of the particle among the states.

** In the general case, one more transformation of the type (3) is needed for the diagonalization of H_{11}. It corresponds to the separation of a self-consistent field of the new quasi-particles α and β.

$$H_{20} = \sum_{\nu} \left\{ (\tilde{\varepsilon}_\nu - \lambda)\, 2\, U_\nu V_\nu - \Delta_\nu (U_\nu^2 - V_\nu^2) \right\} (\alpha_\nu^+ \beta_\nu^+ + \beta_\nu \alpha_\nu), \tag{23}$$

$$H_{11} = \sum_{\nu} \left\{ (\tilde{\varepsilon}_\nu - \lambda)(U_\nu^2 - V_\nu^2) + \Delta_\nu\, 2\, U_\nu V_\nu \right\} (\alpha_\nu^+ \alpha_\nu + \beta_\nu^+ \beta_\nu), \tag{24}$$

where $\tilde{\varepsilon}_\nu$ and Δ_ν are diagonal terms of (19) and (20),

The condition $H_{20} = 0$, which determines U_ν and V_ν, has now the form

$$(\tilde{\varepsilon}_\nu - \lambda)\, 2\, U_\nu V_\nu - \Delta_\nu (U_\nu^2 - V_\nu^2) = 0. \tag{25}$$

4. Analysis of the Equation for $U_\nu V_\nu$

It is convenient to use an alternative form of equation (25). To obtain this, we express U_ν and V_ν through Δ_ν from (25) and (9):

$$\left.\begin{aligned} 2\, U_\nu V_\nu &= \frac{\Delta_\nu}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta_\nu^2}}; \qquad & U_\nu^2 - V_\nu^2 &= \frac{\tilde{\varepsilon}_\nu - \lambda}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta_\nu^2}} \\ U_\nu^2 &= \frac{1}{2}\left[1 + \frac{\tilde{\varepsilon}_\nu - \lambda}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta_\nu^2}}\right]; \qquad & V_\nu^2 &= \frac{1}{2}\left[1 - \frac{\tilde{\varepsilon}_\nu - \lambda}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta_\nu^2}}\right]. \end{aligned}\right\} \tag{26}$$

Using (26) and (20), we find the following equation for Δ_ν:

$$\Delta_\nu = \frac{1}{2} \sum_{\nu'} \frac{\langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle}{\sqrt{(\tilde{\varepsilon}_{\nu'} - \lambda)^2 + \Delta_{\nu'}^2}} \Delta_{\nu'}. \tag{27}$$

To make clear the physical sense of the quantity Δ_ν, let us consider the energy of a quasi-particle E_ν. From (24) and (26) follows

$$E_\nu = \sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta_\nu^2}. \tag{28}$$

As is seen from (28), in the case of a continuous spectrum, the quantity Δ_ν is an energy gap in the spectrum of the quasi-particles. For a discrete spectrum $\tilde{\varepsilon}_\nu$, it is meaningful to speak of a gap only for values of Δ_ν which are greater than the distances between the levels $\tilde{\varepsilon}_\nu$.

The equation (27) has a trivial solution:

$$\Delta_\nu = 0 \quad \text{or} \quad U_\nu V_\nu = 0, \tag{29}$$

which corresponds to the sharp Fermi surface. If we choose in this case

$$\left.\begin{aligned} U_\nu = 1;\ V_\nu = 0 \quad \text{for} \quad \tilde{\varepsilon}_\nu > \lambda, \\ U_\nu = 0;\ V_\nu = 1 \quad \text{for} \quad \tilde{\varepsilon}_\nu < \lambda, \end{aligned}\right\} \qquad (30)$$

then the new quasi-particles α, β should correspond to the old particles outside the Fermi sea and to the old holes inside. If the interaction is sufficiently weak, the trivial solution (30) remains the only solution of (27).

However, if the inequality

$$\frac{1}{2}\sum_{\nu'} \frac{\langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle}{|\, \tilde{\varepsilon}_{\nu'} - \lambda \,|} > 1 \qquad (31)$$

is fulfilled, then there is also a non-trivial solution of (27), which corresponds to the modification of the Fermi sea (the analogy of the superconducting state)*.

The equation (27) contains two matrix elements of the two-body potential which play entirely different roles. The matrix element $\langle \nu\nu' | G | \nu'\nu \rangle$ is shown from (19) to contribute only to the self-consistent field. The matrix element $\langle \nu\nu | G | \nu'\nu' \rangle$ ("pairing interaction"), on the contrary, determines a qualitatively new effect which corresponds to a modification of the Fermi sea. From (31) one can see that this modification is possible only if the pairing interaction for sufficiently many states has a coherent character, e. g., for a sufficiently broad region of the states the matrix element $\langle \nu\nu | G | \nu'\nu' \rangle$ must have the same sign, because otherwise there will occur a cancellation.

Lets us expand the two-body interaction potential in spherical harmonics

$$G(\vec{r}_1 - \vec{r}_2) = \sum_l G^l(r_1 r_2) P_l(\cos\vartheta_{12}). \qquad (32)$$

and ask the question: "Which part of the two-body interaction contributes to a self-consistent field and which part determines a coherent pairing interaction?"

We believe that the following considerations may provide a qualitative understanding of this point. Assume that the spherically symmetric part of the interparticle interaction has determined a certain self-consistent isotropic field. The single-particle levels in this field are degenerate and characterized by the value of the angular momentum j (shell model). Let us consider the particles in the same level j, neglecting their interaction

* Eq. (27) differs from the analogous equation in ref. 7 for a superconductor by the character of the spectrum $\tilde{\varepsilon}_\nu$. In the case of the superconductor, the continuous spectrum allows a non-trivial solution of (27) for any value of the interaction.

with the particles in the other shells. The term with $l = 2$ (quadrupole) gives an essential contribution to a self-consistent field producing an ellipsoidal deformation which splits the single-particle levels[20]. But its contribution to the pairing interaction is small, because it connects only the nearest levels ($|\nu - \nu'| \leqslant 2$), which might not be enough to satisfy the inequality (31). The term in (32) with $l = 4$ connects the more distant levels ($|\nu - \nu'| \leqslant 4$), but its contribution to the self-consistent field is not so important, and so on. Therefore, the main contribution to the pairing interaction is from the high harmonics of the two-particle potential. The self-consistent field, on the other hand, is essentially determined by the low harmonics.

II. Ground State and Single-Particle Excitations

1. Solution of the Equation for Δ_ν

We assume that the condition (31) is fulfilled and that a non-trivial solution of the equation (27) exists. For an explicit solution of (27), assumptions have to be made about the character of the single-particle spectrum $\tilde{\varepsilon}_\nu$. For strongly deformed nuclei, where the shell structure almost completely vanishes, the distribution of the single-particle levels is approximately uniform in each interval, and the average level density is a smooth function of the energy. The sum in (27) spreads practically only over an effective region of the coherent interaction where the matrix element $\langle \nu \nu | G | \nu' \nu' \rangle$ differs appreciably from zero. The single-particle levels of spherical and not strongly deformed nuclei exhibit a shell structure*, i. e., are divided into sharply separated groups[11]. The most essential contribution to the sum (27) is given, in this case, by transitions between the states in the same shell. Neglecting the transitions between different shells (which will be discussed later), we can treat each shell independently. Therefore, in both cases, we have to consider a separated group of levels with approximately uniform distribution. For strongly deformed nuclei, this level group, determined by the effective region of interaction, may be assumed to be symmetrical with respect to the Fermi surface. In the second case, the level group coincides with the shell and may, in particular, reduce to one highly degenerate level. The position of the Fermi energy, in this case, is not fixed and depends on the number of particles in the shell. (A symmetrical position corresponds

* We do not necessarily here mean j-shells.

approximately to a half-filled shell). We shall consider this general case, keeping in mind that the case of strongly deformed nuclei is equivalent simply to a half-filled shell.

To simplify the problem, we assume that the matrix element $\langle \nu \nu | G | \nu' \nu' \rangle$ is constant for the transitions between any levels inside the shell. With this assumption, the equation (27) for Δ (which is now constant) takes the form

$$1 = \tfrac{1}{2} G \sum_{\nu} \frac{1}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta^2}}. \tag{33}$$

For Δ larger than the distance between levels, which is the case we are interested in, the sum in (33) can be replaced by an integral. Then, we get

$$1 = \frac{G}{2} \int_a^b \varrho(\varepsilon) \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \tag{34}$$

where

$$a = \varepsilon' - \lambda; \quad b = \varepsilon'' - \lambda \tag{35}$$

and ε', ε'' determine the boundaries of the shell.

Defining a certain average level density $\bar{\varrho}$, according to

$$\bar{\varrho} \int_a^b \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} = \int_a^b \frac{\varrho(\varepsilon)\, d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}}, \tag{36}$$

and introducing the dimensionless quantity

$$\eta = (\bar{\varrho} G)^{-1}, \tag{37}$$

we obtain from (34)

$$\sinh^{-1} \frac{b}{\Delta} - \sinh^{-1} \frac{a}{\Delta} = 2\eta \tag{38}$$

and, therefore,

$$\Delta = \frac{1}{\sinh 2\eta} [b^2 + a^2 - 2ab \cosh 2\eta]^{1/2}. \tag{39}$$

2. Influence of the neighbouring shells

Up to now, we have not taken into account the matrix elements $\langle \nu \nu | G | \nu' \nu' \rangle$ between different shells. Here, we consider briefly their effect. By separating out those terms of the sum (27), for which ν' is in the shell nearest to the Fermi surface (λ-shell), we can rewrite the equation (27) in the following form:

$$\Delta_\nu = \frac{1}{2}\sum_{\nu'}{}^{\lambda} \frac{\langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle}{\sqrt{(\tilde{\varepsilon}_{\nu'} - \lambda)^2 + \Delta_{\nu'}^2}} \Delta_{\nu'} + \frac{1}{2}\sum_{\nu'}{}' \frac{\langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle}{\sqrt{(\tilde{\varepsilon}_{\nu'} - \lambda)^2 + \Delta_{\nu'}^2}} \Delta_{\nu'}. \tag{40}$$

Treating the last term as a perturbation, we may in this term replace $\Delta_{\nu'}$ by Δ_ν, determined from the main part of (40). Further, in the denominator of this term, Δ_ν^2 can be neglected, since the distance between shells is greater than the gap*. The equation (55) may then be expressed as

$$\left.\begin{aligned} \Delta_\nu = \frac{1}{2}\sum_{\nu'}^{\lambda} \frac{\Delta_{\nu'}}{\sqrt{(\tilde{\varepsilon}_{\nu'} - \lambda)^2 + \Delta_{\nu'}^2}} \Bigg\{ & \langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle \\ + \frac{1}{2}\sum_{\nu''}{}' & \frac{\langle \nu\nu \,|\, G \,|\, \nu''\nu'' \rangle \langle \nu''\nu'' \,|\, G \,|\, \nu'\nu' \rangle}{|\, \tilde{\varepsilon}_{\nu''} - \lambda \,|} \Bigg\}. \end{aligned}\right\} \tag{41}$$

Eq. (41) has the same form as the corresponding equation in which intershell transitions are neglected, but with a new value of the interaction. Therefore, the influence of other shells only increases the effective interparticle interaction.

The efficiency of the pairing interaction depends on the region of interaction, i.e., on the number of states connected by the transitions $\langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle$, and on the value of the matrix element. Inclusion of intershell transitions means, in fact, some extension of the interaction region, but, as it has been shown just now, it can be described as an effective increase in the value of the matrix element. It is of interest to see how this solution for Δ, in the case when the shell structure disappears, goes into solution with the unrenormalized matrix element, but with an extended region of interaction. For simplicity, we consider the case of a uniform level density inside the shells and assume a constant matrix element for all transitions in an energy region $(-\omega, \omega)$ measured from the Fermi surface λ. Then, we have for the last term in (40)

$$\frac{1}{2}\Delta\left(\int_{-\omega}^{a-\delta} + \int_{b+\delta}^{\omega}\right) \frac{G\varrho\, d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}},$$

where δ is the distance between the shells. Neglecting the quantity Δ in the square root, and performing the integration, we obtain for the equation (40)

* This is, in fact, the condition for the existence of a shell structure in our treatment.

$$\sinh^{-1}\frac{b}{\Delta} - \sinh^{-1}\frac{a}{\Delta} = 2\eta - ln\frac{\omega^2}{(b+\delta)(\delta-a)} \equiv 2\,\eta_{\text{eff}}.$$

In the limiting case of $\eta_{\text{eff}} \gg 1$, one finds for Δ

$$\Delta = 2\sqrt{-ab}\,e^{-\eta_{\text{eff}}} = 2\,\omega\sqrt{\frac{ab}{(b+\delta)(a-\delta)}}\,e^{-\eta}$$

which, for $\delta \to 0$, goes into the usual solution for a system with uniform level density (Fermi gas).

The matrix element $\langle \nu\nu | G | \nu'\nu' \rangle$ decreases with the atomic number as A^{-1}. On the other hand, in heavy nuclei, the distance between the shells decreases and intershell transitions become more essential. Therefore, the effective interaction parameter G decreases somewhat more slowly than A^{-1}.

3. Elimination of the Chemical Potential

The operator for the total number of particles in the system has the form

$$\left.\begin{aligned} N = \sum_{k\sigma} a_{k\sigma}^{+} a_{k\sigma} = \sum_{\nu} 2\,V_\nu^2 + \sum_{\nu} (U_\nu^2 - V_\nu^2)(\alpha_\nu^{+}\alpha_\nu + \beta_\nu^{+}\beta_\nu) \\ + \sum_{\nu} 2\,U_\nu V_\nu (\alpha_\nu^{+}\beta_\nu^{+} + \beta_\nu \alpha_\nu). \end{aligned}\right\} \quad (42)$$

The average number of particles in the ground state ("vacuum") is determined by the constant term in (42). Comparing it with a given value N, we find the following equation which determines λ:

$$\sum_{\nu}{}' 2\,V_\nu^2 = \sum_{\nu}{}' \left(1 - \frac{\tilde{\varepsilon}_\nu - \lambda}{\sqrt{(\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta^2}}\right) = N \quad (43)$$

or, replacing the sum by an integral,

$$\int_a^b \left(1 - \frac{\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}}\right) \varrho(\varepsilon)\, d\varepsilon = N. \quad (44)$$

We shall approximate the level density $\varrho(\varepsilon)$ by a straight line, taking into account only the first derivative $\varrho'(\varepsilon)$. Introducing two parameters

$$\varrho_0 = \frac{1}{2}[\varrho(b) + \varrho(a)]; \quad \xi = \frac{\varrho(b) - \varrho(a)}{\varrho(b) + \varrho(a)}, \quad (45)$$

we have in this approximation

$$\varrho(\varepsilon) = \varrho_0 \left[1 - \xi \frac{b+a}{b-a} + \frac{2\,\xi}{b-a}\,\varepsilon\right]. \tag{46}$$

The average density ϱ_0 is connected with the total number of pairing states in the shell Ω by the condition

$$\varrho_0\,(\varepsilon'' - \varepsilon') = \Omega. \tag{47}$$

Performing the integration in (44) and inserting for Δ the expression (39), we get

$$\left.\begin{aligned} 1 - \frac{N}{\Omega} - \frac{b+a}{b-a}\tanh\eta - \frac{\xi}{2}\left(1 - \frac{2\,\eta}{\sinh 2\,\eta}\right)\coth\eta \\ + \frac{\xi}{2}\left(\frac{b+a}{b-a}\right)^2\left(1 - \frac{2\,\eta}{\sinh 2\,\eta}\right)\tanh\eta = 0. \end{aligned}\right\} \tag{48}$$

It is convenient to introduce the quantity χ_N according to

$$b + a = (b-a)\,\chi_N \coth\eta = (\varepsilon'' - \varepsilon')\,\chi_N \coth\eta. \tag{49}$$

Inserting (49) into (48), one finds the following equation for χ_N:

$$\chi_N - \tfrac{1}{2}\,\xi\gamma(\eta)\,\chi_N^2 = 1 - \frac{N}{\Omega} - \tfrac{1}{2}\,\xi\gamma(\eta), \tag{50}$$

where

$$\gamma(\eta) = \coth\eta\left(1 - \frac{2\,\eta}{\sinh 2\,\eta}\right). \tag{51}$$

From (50) follows

$$\chi_N = 2\,\frac{1 - \dfrac{N}{\Omega} - \dfrac{\xi}{2}\gamma}{1 + \sqrt{1 - 2\,\xi\gamma\left(1 - \dfrac{N}{\Omega}\right) + \xi^2\gamma^2}}. \tag{52}$$

From (45) and (51), one can see that $|\xi| \leqslant 1$ and $0 \leqslant \gamma < 1$. Thus, from (52) it follows that $|\chi_N| \leqslant 1$. The limiting values ± 1 are reached on the boundaries of the shell ($N = 0$ and $N = 2\Omega$):

$$\chi_N = \left\{\begin{array}{l} 1 \ \text{for}\ N = 0, \\ 0 \ \text{for}\ N = (1 - \frac{1}{2}\,\xi\gamma), \\ -1 \ \text{for}\ N = 2\,\Omega. \end{array}\right\} \tag{53}$$

Thus, the quantity χ_N characterizes the occupation of the shell and may be called "occupation factor". For the uniform level density ($\xi = 0$),

$$\chi_N \to \chi_N^0 = 1 - \frac{N}{\Omega}. \tag{54}$$

The chemical potential λ may be expressed from (49) and (35) as

$$\lambda = \frac{\varepsilon'' + \varepsilon'}{2} - \frac{1}{2}(\varepsilon'' - \varepsilon')\,\chi_N \coth\eta. \tag{55}$$

The average density $\bar{\varrho}$ introduced in (36) (and therefore η) in general depends on N. Inserting (46) into (36) and performing the integration, we find

$$\bar{\varrho} = \varrho_0\left[1 - \xi\chi_N\left(\coth\eta - \frac{1}{\eta}\right)\right]. \tag{56}$$

With the aid of (55) or (49) λ can be eliminated from all final results.

4. Criterion for the Existence of a Gap

Eliminating λ from (39) we get for Δ*

$$\Delta = \frac{\varepsilon'' - \varepsilon'}{2\sinh\eta}(1 - \chi_N^2)^{1/2}\,{}^{**}. \tag{57}$$

This result has been obtained with the assumption that Δ is not smaller than the distance between levels. This condition may be written as $\Delta\varrho_0 > 1$ or, with the aid of (57) and (47), as

$$\frac{\Omega}{2\sinh\eta}(1 - \chi_N^2)^{1/2} > 1. \tag{58}$$

This inequality gives the condition for a modification of the Fermi sea and for the existence of an energy gap for the nucleus. To estimate the left side of (58), we may use the expression (54) for χ_N and obtain

$$\frac{1}{2\sinh\eta}\sqrt{N(2\Omega - N)} > 1. \tag{59}$$

* We avoid calling Δ an "energy gap". It will be seen later, that in the case $\eta < 1$, the energy gap is determined by another quantity.

** Far from the boundaries of the shell, where $\chi_N^2 \ll 1$, (57) coincides with the solution in references 6, 7 for a superconductor if $(\varepsilon'' - \varepsilon')$ is identified with the region of the coherent interaction (see also pages 14–15).

In the case of $\eta \gg 1$, (59) is satisfied only by a sufficiently broad region of coherent interaction, i. e., for sufficiently large values of $N(2\Omega - N)$. On the other hand, if $\eta \lesssim 1$, this condition is always fulfilled.

5. Energy of the Ground State

The energy of the ground state (of the "vacuum") is the quantity U given by (22). The last term in (22) is connected with the energy of the selfconsistent field. We shall come to this term later and consider here only the two first terms in (22), i. e.,

$$U' = \sum_{\nu} (\tilde{\varepsilon}_{\nu} - \lambda)\, 2 V_{\nu}^2 - \sum_{\nu} \Delta_{\nu} U_{\nu} V_{\nu}. \tag{60}$$

The sum over closed shells, for which $\Delta_{\nu} = 0;\ V_{\nu} = 1$, gives

$$U'_{cl} = 2 \sum_{\nu}{}' (\tilde{\varepsilon}_{\nu} - \lambda) = 2 \sum_{\nu}{}' \tilde{\varepsilon}_{\nu} - \lambda N_{cl}. \tag{61}$$

Using the constancy of Δ_{ν} and replacing the sum by an integral, we get for the unfilled λ-shell

$$U'_{\lambda} = \int_a^b \left(1 - \frac{\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}}\right) \varrho(\varepsilon)\, \varepsilon\, d\varepsilon - \frac{\Delta^2}{G}.$$

After the integration and subsequent elimination of λ with the help of (49), we find

$$U'_{\lambda} = -\frac{\Omega}{4} (\varepsilon'' - \varepsilon')(1 - \chi_N)^2 \left[\coth \eta - \frac{\xi}{3}(2 + \chi_N)\right]. \tag{62}$$

The quantity U corresponds to the ground state of the auxiliary Hamiltonian (2) with the chemical potential. The energy which corresponds to the original Hamiltonian according to (62) and (55) is given by

$$\left.\begin{aligned} W_{\lambda} = U'_{\lambda} + \lambda N_{\lambda} = \frac{1}{2}\left[\varepsilon' + \varepsilon'' + \frac{\xi}{3}(\varepsilon'' - \varepsilon')\right] N_{\lambda} \\ -\frac{\Omega}{4}(\varepsilon'' - \varepsilon')(1 - \chi_N^2)\left[\coth \eta + \frac{\xi}{3}\chi_N (1 - 3\gamma(\eta)\coth \eta) - \frac{\xi^2}{3}\gamma(\eta)\right]. \end{aligned}\right\} \tag{63}$$

The last term in (63) contains the factor $(1 - \chi_N^2)$ and disappears for the closed shell, which has, thus, the energy

$$W_{cl} = \frac{1}{2}\left[\varepsilon' + \varepsilon'' + \frac{\xi}{3}(\varepsilon'' - \varepsilon')\right] 2\,\Omega. \tag{64}$$

It is known that the energy of the closed shell in the first approximation does not change for a small variation $\delta\beta$ of the equilibrium deformation. Therefore, the quantity

$$\varepsilon_\lambda = \frac{1}{2}\left[\varepsilon' + \varepsilon'' + \frac{\xi}{3}(\varepsilon'' - \varepsilon')\right]$$

is at least quadratic in $\delta\beta$*.

In the absence of the interaction ($\eta \to \infty$; $\gamma \to 1$), it follows from (63) that

$$W_\lambda = \varepsilon_\lambda N - \frac{\Omega}{4}(\varepsilon'' - \varepsilon')(1 - \chi_N^2)\left[1 - \tfrac{2}{3}\xi\chi_N - \tfrac{1}{3}\xi^2\right]. \tag{65}$$

In the opposite limiting case $\eta < 1$, one finds

$$\left.\begin{aligned} &\gamma \approx \frac{2}{3}\eta; \quad \chi_N \approx \chi_N^0 - \frac{\xi}{3}\Theta_N\eta - \frac{2}{3}\xi^2\chi_N^0\Theta_N\eta^2 \\ &\varepsilon'' - \varepsilon' \approx \Omega G\eta\left[1 - \frac{\xi}{3}\chi_N^0\eta + \frac{\xi^2}{g}\Theta_N\eta^2\right], \end{aligned}\right\} \tag{66}$$

where χ_N^0 is given by (54), and

$$\Theta_N = 1 - (\chi_N^0)^2 = \frac{2N}{\Omega}\left(1 - \frac{N}{2\Omega}\right). \tag{67}$$

Inserting (66) in (63) and restricting ourselves to η^2-terms, we find

$$W_\lambda = \varepsilon_\lambda N - \frac{1}{4}\Omega^2 G\Theta_N - \frac{\Omega^2}{12}G\Theta_N\left(1 - \frac{\xi^2}{3}\Theta_N\right)\eta^2. \tag{68}$$

For small deformations β of a spherical nucleus, $\eta \approx (\varepsilon'' - \varepsilon')/\Omega G \sim \beta$. The expression (65) is thus a linear function of β. On the other hand, (68) is proportional to β^2. Therefore, the pairing interaction changes the dependence of the energy of the outside nucleons on deformation. This turns out to be very important for the problem of the nuclear equilibrium shape. (See Part III).

6. Energy Spectrum of Quasi-Particles

The ground state of the system expressed in terms of the quasi-particles $\alpha\beta$ is a vacuum. Acting on the vacuum wave function by the operators α^+ and β^+, we get excited states with one or more quasi-particles. Such states we shall denote as single-particle excited states. The energy of these

* For a correct approximation the quantity ξ must be chosen to satisfy this condition.

states measured relative to the ground state is given by the sum of the quasi-particle energies E_ν.

From (28), (55), and (57) we find for E_ν

$$\left.\begin{aligned} E_\nu^2 &= (\tilde{\varepsilon}_\nu - \lambda)^2 + \Delta^2 \\ &= \frac{1}{4}(\varepsilon'' - \varepsilon')^2 (1 - \chi_N^2)(\coth^2\eta - 1) + \left[\tilde{\varepsilon}_\nu - \frac{\varepsilon' + \varepsilon''}{2} + \frac{1}{2}(\varepsilon'' - \varepsilon')\chi_N \coth\eta\right]^2. \end{aligned}\right\} \quad (69)$$

In the absence of a pairing interaction, E_ν coincides with the particle energy measured relative to the Fermi surface:

$$E_\nu^0 = |\tilde{\varepsilon}_\nu - \lambda^0| = \pm(\tilde{\varepsilon}_\nu - \lambda_0) = \pm\left\{\tilde{\varepsilon}_\nu - \tfrac{1}{2}(\varepsilon'' + \varepsilon') + \tfrac{1}{2}(\varepsilon'' - \varepsilon')\bar{\chi}_N\right\}, \quad (70)$$

where $\bar{\chi}_N = \chi_N(\eta = \infty)$. With the aid of (70), the expression (69) takes the form

$$\left.\begin{aligned} E_\nu^2 = \frac{1}{4}(\varepsilon'' - \varepsilon')^2 \Big\{ & \coth^2\eta - 1 + (\chi_N - \bar{\chi}_N)^2 - 2\,\chi_N\bar{\chi}_N(\coth\eta - 1) \\ & \pm \frac{4\,E_\nu^0}{\varepsilon'' - \varepsilon'}(\chi_N \coth\eta - \bar{\chi}_N) + \frac{4\,E_\nu^{02}}{(\varepsilon'' - \varepsilon')^2}\Big\}. \end{aligned}\right\} \quad (71)$$

In the limit of $\eta \gg 1$ (weak interaction or small level density), we get

$$E_\nu \approx \sqrt{(\varepsilon'' - \varepsilon')^2(1 - \chi_N^2)\,e^{-2\eta} + E_\nu^{02}} = \sqrt{\Delta^2 + E_\nu^{02}}. \quad (72)$$

In this case, the value of the energy gap is given by the quantity Δ. Because of the factor $(1 - \chi_N^2)$ the gap disappears, in this case, at closed shells.

In the opposite limiting case of $\eta < 1$, it follows from (71) that

$$E_\nu = \sqrt{\tfrac{1}{4}\,\Omega^2 G^2 \pm \chi_N^0\,\Omega G E_\nu^0 + E_\nu^{02}}. \quad (73)$$

In (73) the role of the gap is played by the quantity $^1/_2\,\Omega G$. It does not depend on the number of particles in the shell and does not coincide with Δ which, for $\eta \ll 1$, is equal to

$$\Delta \approx \frac{1}{2}\Omega G(1 - (\chi_N^0)^2)^{1/2} = \frac{1}{2}\Omega G \Theta_N^{1/2}. \quad (74)$$

For the first excited states, E_ν^0 is of the order of the distance between single-particle levels (in the absence of degeneracy $E_\nu^0 \sim \varrho^{-1}$). Therefore, the terms in (73) containing E_ν^0 are small. These terms determine the level density of the quasi-particles above the gap. For small E_ν^0, we have from (73)

$$E_\nu \approx \frac{1}{2}\Omega G \pm \left(\chi_N^0 \pm \frac{E_\nu^0}{\Omega G}\right) E_\nu^0. \quad (75)$$

(The term, quadratic in E_ν^0, may be important only in the middle of the shell, where $\chi_N^0 \approx 0$). Comparing the level density ϱ for the quasi-particles with that for the original non-interacting particles ϱ^0, one finds

$$\varrho^0/\varrho \approx \chi_N^0 + \frac{2E_\nu^0}{\Omega G} = 1 - \frac{N}{\Omega} \pm \frac{2E_\nu^0}{\Omega G}. \tag{76}$$

Near the closed shell, where $|\chi_N^0| \sim 1$, the ratio (76) is of the order of unity. As one moves away from closed shells, the level density of quasi-particles increases and in the middle of the shell becomes of the order of $\Omega\varrho^0$.

7. Wave Functions of the Ground- and Excited States

We consider now the question of the meaning of the wave functions of the ground- and excited states in terms of the old particles. Let us introduce the wave function of the vacuum state of the old particles $\Psi_0^{(0)}$, for which $a_{k\sigma}\Psi_0^{(0)} = 0$. The first transformation (3), which removes the self-consistent field, does not mix up creation and destruction operators and, therefore, does not change the vacuum state. After introduction of the operators α_ν and β_ν, the new vacuum state Ψ_0 is defined by the equations

$$\left.\begin{aligned} \alpha_\nu \Psi_0 &= (U_\nu b_{\nu+} - V_\nu b_{\nu-}^+)\Psi_0 = 0, \\ \beta_\nu \Psi_0 &= (U_\nu b_{\nu-} + V_\nu b_{\nu+}^+)\Psi_0 = 0. \end{aligned}\right\} \tag{77}$$

It is easy to prove that these equations are satisfied by the function

$$\Psi_0 = \prod_\nu (U_\nu + V_\nu b_{\nu+}^+ b_{\nu-}^+)\Psi_0 = 0. \tag{78}$$

In this representation, the wave functions of the excited states with only one quasi-particle have the form

$$\left.\begin{aligned} \alpha_\nu^+ \Psi_0 &= \prod_{\nu' \neq \nu} (U_{\nu'} + V_{\nu'} b_{\nu'+}^+ b_{\nu'-}^+) b_{\nu+}^+ \Psi_0^{(0)}, \\ \beta_\nu^+ \Psi_0 &= \prod_{\nu' \neq \nu} (U_{\nu'} + V_{\nu'} b_{\nu'+}^+ b_{\nu'-}^+) b_{\nu-}^+ \Psi_0^{(0)}, \end{aligned}\right\} \tag{79}$$

and the function corresponding to the excitation of the pair is

$$\alpha_\nu^+ \beta_\nu^+ \Psi_0 = \prod_{\nu' \neq \nu} (U_{\nu'} + V_{\nu'} b_{\nu'+}^+ b_{\nu'-}^+)(U_\nu b_{\nu+}^+ b_{\nu-}^+ - V_\nu)\Psi_0^{(0)}\,*. \tag{80}$$

* The expressions (78)—(80) are similar to those obtained in reference 6 for a superconductor.

As it follows from (78), Ψ_0 describes a superposition of states with different numbers of particles. This is true also for the functions (79) and (80). It must be pointed out that (78) and (80) are formed only by states with even numbers of particles. The functions (79), on the contrary, describe only superpositions of states with odd N. Therefore, these functions belong to different physical systems.

The "vacuum" function Ψ_0 describes the ground state only of the even-N system (even-even nuclei). Excited states in such systems contain an even number of quasi-particles α^+ or β^+ and are separated from the ground state by twice the energy gap.

For the odd-N systems, the ground state is given by the lowest of the states (79), i. e., the state with one quasi-particle, say $\alpha^+_{\nu_0}\,\Psi_0$. The excitations of this odd quasi-particle, which are obtained by acting with the operators $\alpha^+_\nu\,\alpha_{\nu_0}$ or $\beta^+_\nu\,\alpha_{\nu_0}$, have no energy gap.

Therefore, the excitation spectra in even-even and odd nuclei turn out to be completely different. On the other hand, the properties connected with the energy of the ground state exhibit no essential differences, since the energy of the odd particle may be neglected with comparison to the "vacuum" energy.

8. Uncertainty in the Number of Particles

To estimate the uncertainty in the number of particles in the states (78)—(80), one can consider the average quadratic fluctuation of N, say, in the "vacuum" state Ψ_0. Using for N the expression (42), one easily finds

$$\langle N^2\rangle - \langle N\rangle^2 = \sum_\nu 4\,U_\nu^2 V_\nu^2 = \sum_\nu \frac{\Delta_\nu^2}{\Delta_\nu^2 + (\tilde{\varepsilon}_\nu - \lambda)^2}. \tag{81}$$

For simplicity, we restrict ourselves to the case of a uniform level density ($\xi = 0$). Replacing the sum in (81) by an integral, we find, after some elementary calculations,

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{\Omega}{\sinh\eta}\,\Theta_N^{1/2}\tan^{-1}(\Theta_N^{1/2}\sinh\eta), \tag{82}$$

where Θ_N is the occupation factor (67). In the limiting case of $\eta \gg 1$, we get

$$\langle N^2\rangle - \langle N\rangle^2 \approx \pi\,\frac{\Omega}{2\sinh\eta}\,\Theta_N^{1/2} \qquad (\eta \gg 1). \tag{83}$$

This expression differs from the left side of the inequality (58) only by the factor π. Therefore, for this case, we can write

$$\langle N^2\rangle - \langle N\rangle^2 \approx \pi \varrho \Delta \qquad (\eta \gg 1). \tag{84}$$

For strongly deformed nuclei for which this case is realized, the value $\varrho\Delta$ is significantly smaller than N, the number of the particles in the un closed shell.

In the case of $\eta < 1$, (82) takes the form

$$\langle N^2\rangle - \langle N\rangle^2 \approx \Omega\,\Theta_N = 2\,N\left(1 - \frac{N}{2\,\Omega}\right); \quad (\eta < 1). \tag{85}$$

For $N = 2$ (one pair), δN, the average width of the distribution is approximately 2, i. e., there are admixed practically only the nearest even neighbours. In the middle of the shell ($N \sim \Omega$) the width is of the order of $\sqrt{N}$.

One might suspect that the uncertainty of N, in spite of its smallness, is of principal importance, because it might permit solutions which are impossible for fixed N. It must be pointed out that the removal of the condition $N =$ const by the introduction of the chemical potential λ does not extend the scope of possible solutions. This method means only a replacement of the system under consideration (N-system by λ-system). There are no physical reasons to expect a significant change in the ground state and in the properties of quasi-particles caused by this replacement. We can see this in the limiting case $\eta = 0$ (complete degeneracy), where our results may be compared with the exact solution*. The energy spectrum of the system with N particles is given in this case by[12].

$$W_m^N = -\frac{1}{2}\Omega G N\left(1 - \frac{N-2}{2\,\Omega}\right) + \Omega G m\left(1 - \frac{m-1}{\Omega}\right). \tag{86}$$

The corresponding expression in our case (λ-system) follows from (68) and (73) by substituting $\eta = 0$, $E_\nu^0 = 0$, $\varepsilon_\nu = 0$ by

$$W_m^\lambda = -\frac{1}{2}\,\Omega G N\left(1 - \frac{N}{\Omega}\right) + \Omega G m \tag{87}$$

(the last term corresponds to m excited pairs). Comparing (86) and (87) it is seen that relative corrections both to the ground state and to excited states are of the order of Ω^{-1} **.

* The Hamiltonian is given by $H = -G\sum_{\nu\nu'} b^+_{\nu+}\, b^+_{\nu-}\, b_{\nu'-}\, b_{\nu'+}$

** The interaction, H_{int}, between quasi-particles gives corrections of the same order. Therefore, in the approximation of independent quasi-particles, the equations (87) and (86) do not differ from each other.

9. Effect of the Residual Interaction between Quasi-Particles

The nature of the canonical transformations performed above might be explained in the following way. The interaction between the original particles contains a coherent pairing energy. This interaction, in principle, could be treated in a direct way by rejecting of the independent-particle model. We had another aim, namely to keep this model, but to take into account the pairing interaction, or at least its main part, by introducing a new type of independent particles. The pairing energy, which was an interparticle interaction, then determines the intrinsic structure of the quasi-particles.

With the aid of the canonical transformations, we can take into account the pairing interaction only in the form of the matrix elements $\langle \nu\nu|G|\nu'\nu'\rangle$. The question might quite naturally arise as to whether these matrix elements are the main part of the pairing interaction. Other matrix elements could possibly cause results which are basically different.

The residual interaction between the new quasi-particles is described by the Hamiltonian H_{int} (16). To answer the questions mentioned above one might treat H_{int} as a perturbation. In our case, the perturbation treatment has a special feature, since the coefficients of the canonical transformation U_ν, V_ν have to be corrected in each order. In the second order in H_{int}, the structure of the equation (27) for Δ_ν does not change, but the matrix element $\langle \nu\nu|G|\nu'\nu'\rangle$ is replaced by an expression of the form*

$$\langle \nu\nu \mid G \mid \nu'\nu'\rangle + \sum_{\nu_1\nu_1'} \frac{\langle \nu\nu_1 \mid G \mid \nu_1'\nu'\rangle \langle \nu_1\nu \mid G \mid \nu'\nu_1'\rangle}{E_\nu + E_{\nu'} + E_{\nu_1} + E^{\nu_1'}} U_{\nu_1} V_{\nu_1} U_{\nu_1'} V_{\nu_1'}.$$
$$+ \text{ terms of similar form}$$

In other words, the graphs of the perturbation theory correct the pairing interaction. One can expect that the sum of the graphs would lead to replacing the matrix element $\langle \nu\nu \mid G \mid \nu'\nu'\rangle$ by a certain effective pairing interaction, but not to a basic change of the results**. In this sense, the influence of the residual interaction H_{int} on the properties of the ground state and the quasi-particles is not essential.

* See the analogous analysis for a superconductor in reference 13.

** The analogous procedure in the Brueckner-Bethe theory of nuclear matter [14] leads to a replacement of the interaction matrix elements by those of the transition matrix. A similar situation might be pointed out also in the theory of superfluidity in a Bose system. In reference 15, a spectrum of quasi-particles has been obtained with the aid of a canonical tranformation. The sum of graphs of the perturbation theory, which has been performed in ref. 16, gave the same result but with replacement of the interaction matrix elements by exact scattering amplitudes.

Besides that, the interaction between quasi-particles contains the low harmonics of the nucleon-nucleon interaction (32) which remains almost untouched by the canonical transformation. These harmonics, which give rise to collective excitations in nuclei, require other methods of consideration[20]. On the other hand, collective excitations in nuclei can be treated directly in the framework of the unified nuclear model by introducing a time-dependent deformed self-consistent field.

III. Collective Excitations in Nuclei

The nature of collective excitations in nuclei and the methods of their investigation are explained in detail in the literature (see, e. g., Chapter V of reference 3). We briefly sketch some essential points which will be needed later.

Let us introduce a parameter which describes a particular type of the collective motion. Using the adiabatic character of the collective motion one may first consider the intrinsic motion of the nucleus for a fixed value of ϑ. The energy eigenvalues for this motion are denoted by $W_i(\vartheta)$. Then, the Hamiltonian of the collective motion is given approximately by

$$H_{\text{coll}} = W_i(\vartheta) + \frac{1}{2} B_i(\dot{\vartheta})^2, \tag{88}$$

where the inertial parameter $B_i(\vartheta)$, obtained by the adiabatical perturbation theory, is given by[17]

$$B_i(\vartheta) = 2\hbar^2 \sum_{j \neq i} \frac{\left|\left\langle j \left| \frac{\partial}{\partial \vartheta} \right| i \right\rangle\right|^2}{W_j - W_i}. \tag{89}$$

The potential energy of the collective motion $W_i(\vartheta)$ and the inertial parameter $B_i(\vartheta)$ are essentialy determined by the intrisic nucleon motion and their calculation is possible, in practice, only for simple models. It is known that the hydrodynamical model of irrotational flow gives too small a value for $B(\vartheta)$. The independent-particle model (using an oscillator potential) leads to a very large value of $B(\vartheta)$ for rotations (rigid-body moment of inertia) and to a very small $B(\vartheta)$ for vibrations, which violates the adiabatic condition.

Below, the parameters of the collective motion will be found for the model of independent quasi-particles (which is equivalent to a model of the old particles with the pairing interaction included). It is not our aim to make here a detailed investigation of the collective excitations or a comparison of the results with experimental data. The main problem is to establish what role the pairing interaction plays in collective nucleon motion and what qualitative results it leads to.

1. Dependence of the Nuclear Energy on the Deformation

Here, we restrict ourselves only to the axially symmetric quadrupole deformations. In the liquid drop model, the deformation is defined in a natural way as a deviation of the uniform drop from the spherical shape, and is uniquely connected with the nuclear quadrupole moment*. In single-particle models where one considers the nucleons in a certain potential well, such a simple picture is valid only for nuclei with closed shells. In the presence of particles in an unfilled shell, the nucleus does not behave as a homogeneous system. The nuclear quadrupole moment is not determined only by the deformation of the well, but depends essentially on the configuration of the particles in the unfilled shell. The energy of the nucleus will also depend, in this case, on both factors. However, one must take into account the self-consistent nature of the nuclear potential. Self-consistency requires that the distribution of the potential must be the same as the density distribution (which is the consequence of the short range nucleon-nucleon forces). Therefore, for a given value of the eccentricity of the well, only such configurations of the outside nucleons are allowed, which provide the same eccentricity of the density.

In Section II. 5, we have not used this self-concistency argument; therefore the ground-state energy obtained there applies to the system in an external potential. In order to introduce the self-consistency, we now ask for the lowest state of the system with a fixed value of the quadrupole moment. Due to a relatively small coupling between the closed-shell core and the outside nucleons, we shall consider the deformations of these two components as distinct degrees of freedom and shall be looking for the nuclear energy as a function of two deformation parameters, say, the quadrupole moments both for the closed-shell core and the outside nucleons.

Let us assume, first, that the closed-shell core is spherical and undeformable. In this case, we have to find the energy of the lowest state of the outside nucleons for a fixed value of their quadrupole moment Q_λ (which, in this case, represents the total nuclear moment Q). To satisfy the subsidiary condition of a constancy of Q, we add to the Hamiltonian the term $-\mu\hat{Q}$,

* Here and below, we mean the quadrupole moment of the mass, but not that of the charge.

where $\hat{Q}$ is the quadrupole moment operator, and look for the ground state of the Hamiltonian

$$\overline{H} = H - \mu \hat{Q}. \tag{90}$$

Then, the Lagrangian multiplier μ has to be eliminated by using the condition $\langle \hat{Q} \rangle = Q$.

The quadrupole moment $\hat{Q}$, represented by the sum of the single-particle operators, has the form

$$\left.\begin{aligned} \hat{Q} = \sum_{\nu} q_{\nu\nu} 2 V_\nu^2 + \sum_{\nu\nu'} q_{\nu\nu'} (U_\nu U_{\nu'} - V_\nu V_{\nu'}) (\alpha_\nu^+ \alpha_{\nu'} + \beta_{\nu'}^+ \beta_\nu) \\ + \sum_{\nu\nu'} q_{\nu\nu'} (U_\nu V_{\nu'} + V_\nu U_{\nu'}) (\alpha_\nu^+ \beta_{\nu'}^+ + \beta_\nu \alpha_{\nu'}), \end{aligned}\right\} \tag{91}$$

where $q_{\nu\nu'}$ are the matrix elements of the single-particle quadrupole moment. We neglect, as always, the interaction between quasi-particles and consider instead of H the Hamiltonian $H_0 = U + H_{20} + H_{11}$. Comparing (90) with (22)—(24) one can see that the inclusion of the term $-\mu \hat{Q}$ is simply equivalent to a renormalization of the single-particle energy $\varepsilon_{\nu\nu'} \to (\varepsilon_{\nu\nu'} - \mu q_{\nu\nu'})$. Assuming that the new single-particle energy has been diagonalized by an appropriate choice of the states ν, we get the Hamiltonian of the form (22)—(24), where the levels $\tilde{\varepsilon}_\nu$ are given by

$$\tilde{\varepsilon}_\nu = \varepsilon_\nu - \sum_{\nu'} \langle \nu\nu' \mid \bar{G} \mid \nu'\nu \rangle V_{\nu'}^2 - \mu q_{\nu\nu}. \tag{92}$$

In producing the deformation of nuclei, the quadrupole part of the interaction between particles (the term $l = 2$ in (32)) is of great importance. The main effect of this quadrupole interaction can be described as an interaction of each particle with the total nuclear quadrupole moment. Therefore, we assume that

$$\sum_{\nu'} \langle \nu\nu' \mid \bar{G} \mid \nu'\nu \rangle V_{\nu'}^2 = \varkappa Q q_{\nu\nu}, \tag{93}$$

where $\varkappa$ is a constant coefficient.* **

* In the unified nuclear model, the analogous expression is considered as a coupling energy of a single particle to the nuclear surface. Comparing (93) with the equation (II. 26) of ref. 2 ($W_{\text{coupl}} = -k \beta Y_{20}$), we obtain the following relation between $\varkappa$ and the "coupling constant" k:

$$\varkappa = \frac{5}{12} \frac{k}{A R_0^2 \overline{r^2}}, \tag{93'}$$

where we have used the connection between β and Q (see below, Eq. (105)) and the equation $q = 4 \sqrt{\frac{\pi}{5}} Y_{20} \overline{r^2}$. The equation (93′) implies, in particular, that $\varkappa$ is proportional to $A^{-7/3}$.

** The-single particle energy in a potential well is given by the same equations (92), (93), provided the quantity Q means the quadrupole moment of the potential. Identification of Q with the particle quadropole moment leads to the self-consistency discussed above.

Inserting (93) in (92) one finds for the single-particle-levels

$$\tilde{\varepsilon}_\nu = \varepsilon_\nu - (\mu + \varkappa Q)\, q_{\nu\nu} \equiv \varepsilon_\nu - \tilde{\mu}\, q_{\nu\nu}. \tag{94}$$

The quantity ε_ν, according to (7), is given by $\varepsilon_\nu = \sum_k \varepsilon_k^0 \varphi_{k\nu}^* \varphi_{k\nu}$, where ε_k^0 is the energy of the degenerate single-particle levels in the spherical nucleus, and $\varphi_{k\nu}$ are the coefficients which transform the single-particle wave functions in the spherical field to those in the deformed field. The splitting of ε_ν by the deformation is caused only by the change of the single-particle wave functions and can be neglected for the outside nucleons for which the main splitting is caused by the last term in (94), associated with the direct quadrupole interaction. (Cf. an explicit solution in Appendix B).

The energy of the ground state of the auxiliary Hamiltonian (90) is given by the quantity U in (22). The lowest state with a given value of the quadrupole moment Q of the original Hamiltonian has the energy

$$W(Q) = U + \lambda N + \mu Q. \tag{95}$$

With the aid of (22), (43), and (95) we obtain

$$W(Q) = \sum_\nu \varepsilon_\nu\, 2 V_\nu^2 - \frac{1}{2} \varkappa Q^2 - \frac{\Delta^2}{G}, \tag{96}$$

where Δ is given by (57). The first term in (96) corresponds to the energy of non-interacting particles, while the last two terms represent the energy of the quadrupole and pairing interactions.

According to (91) and (94) the quadrupole moment of the outside nucleons is given by

$$Q_\lambda = \sum_\nu^\lambda q_{\nu\nu}\, 2 V_\nu^2 = \frac{1}{\tilde{\mu}} \sum_\nu^\lambda (\varepsilon_\nu - \tilde{\varepsilon}_\nu)\, 2 V_\nu^2.$$

Neglecting the splitting of ε_ν, we may replace ε_ν by the average energy of the λ-shell $\varepsilon_\lambda = \frac{1}{2}\left[(\varepsilon'' + \varepsilon' + \frac{\xi}{3}(\varepsilon'' - \varepsilon')\right]$. With the aid of (43) we find

$$Q_\lambda = \frac{1}{\tilde{\mu}} (\varepsilon_\lambda - \lambda)\, N - \frac{1}{\tilde{\mu}} \sum_\nu^\lambda (\tilde{\varepsilon}_\nu - \lambda)\, 2 V_\nu^2. \tag{97}$$

Replacing the sum in (97) by an integral we obtain, after calculations similar to those performed in Part II,

$$Q_\lambda = q_0 A(\eta), \tag{98}$$

where

$$q_0 = \max q_{\nu\nu} - \min q_{\nu\nu} = q(\varepsilon') - q(\varepsilon'') = \frac{\varepsilon'' - \varepsilon'}{\tilde{\mu}} \tag{99}$$

is the amplitude of the single-particle quadrupole moment, and

$$A(\eta) = \frac{\Omega}{4}(1-\chi_N^2)\left[\left(1-\frac{\xi^2}{3}\right)\gamma(\eta) - 2\,\xi\chi_N\left(\gamma(\eta)\coth\eta - \frac{2}{3}\right)\right] \tag{100}$$

where the function $\gamma(\eta)$ is given by (51). The equation (98) connects the quadrupole moment of the outside nucleons with the parameter η. Therefore, the equation (96) gives the nuclear energy as a function of η in the case of a spherical undeformable closed-shell core.

Now, let us go to the general case and consider also deformations of the core. Here, we require fixed values of the quadrupole moments both for the outside nucleons and the closed-shell core, and introduce into the Hamiltonian two Lagrangian multipliers

$$\overline{H} = H - \mu\hat{Q}_\lambda - \mu'\hat{Q}_{cl}, \tag{101}$$

where Q_{cl} is the quadrupole moment of the closed shells. The equation (93) is valid also in this case, provided Q means the total quadrupole moment. The expressions (92) and (94) now correspond to the outside nucleons; for the closed shells, one needs simply to replace μ by μ'. After simple calculation, one finds that the energy of the lowest state with given values of Q_λ and Q_{cl} is given by the equation (96), where Q is now the total quadrupole moment ($= Q_{cl} + Q_\lambda$) and the sum in the first term is extended over the closed shells as well as the unfilled λ-shell.

The single-particle quadrupole moment $q_{\nu\nu}$ can be written, for small deformations of the self-consistent field, as

$$q_{\nu\nu} = q_{\nu\nu}^0 + q_{\nu\nu}^{(1)}, \tag{102}$$

where $q_{\nu\nu}^0$ is determined by wave functions in the spherical field, and $q_{\nu\nu}^{(1)}$ is a correction caused by a dependence of the wave functions on the deformation. For the closed shells, $\sum_\nu{}' q_{\nu\nu}^0 = 0$, and therefore Q_{cl} is determined only by the quantity $q_{\nu\nu}^{(1)}$ which is proportional to the deformation. On the other hand, the quantity $q_{\nu\nu}^0$ gives the main contribution to the quadrupole moment of the outside nucleons Q_λ. The contribution of the quantity $q_{\nu\nu}^{(1)}$ to the value of Q_λ turns out to be small, as $\dfrac{\varepsilon''-\varepsilon'}{\varepsilon_F}$, where ε_F is the Fermi energy (see Appendix B). Thus, we can use, for Q_λ, the expression (98), where the dependence of the single-particle wave functions on the deformation was neglected.

The first term in (96) corresponds to the energy of non-interacting particles, provided one considers their new distribution among the levels ($V^2 \neq 0,1$). The dependence of this term on the deformation of the field is caused only by a change in the single-particle wave functions. This leads to a quadratic dependence for small deformations. Choosing the value of the quadrupole moment of the closed-shell core as the deformation parameter, one may write

$$\sum_{\nu} \varepsilon_{\nu} 2 V_{\nu}^2 = W_0 + \frac{1}{2} k Q_{cl}^2, \tag{103}$$

where W_0 and k do not depend on the deformation*. Inserting (103) in (96), one finds

$$W(Q) = W_0 + \frac{1}{2}(k - \varkappa) Q^2 - k Q Q_{\lambda} + \frac{1}{2} k Q_{\lambda}^2 - \frac{\Delta^2}{G}. \tag{104}$$

Let us introduce the deformation β which is associated with the total quadrupole moment Q by the equation[2)]

$$Q = \frac{3}{\sqrt{5\pi}} A R_0^2 \beta \equiv \overline{Q} \beta, \tag{105}$$

where A is the atomic number and R_0 is the nuclear radius. Inserting (98) and (105) in (104), we obtain

$$W(Q) = W_0 + \frac{1}{2}(k - \varkappa) \overline{Q}^2 \beta^2 - k q_0 \overline{Q} \beta \Lambda(\eta) + \frac{1}{2} k q_0^2 \Lambda^2 - \frac{\Delta^2}{G}. \tag{106}$$

The equation (106) determines the nuclear energy as a function of the deformation β and the parameter η associated with the configuration of the outside nucleons.

2. Equilibrium Shape of the Nucleus

In the absence of pairing interaction ($\eta \rightarrow \infty$) one finds from (57) and (100)

$$\frac{\Delta^2}{G} \approx 0 \text{ and } \Lambda(\eta) \approx \frac{\Omega}{4}(1 - \overline{\chi}_N^2)\left[1 - \frac{2}{3}\xi \overline{\chi}_N - \frac{\xi^2}{3}\right], \tag{107}$$

* The quantities W_0 and k depend somewhat on the occupation of the unfilled shell. We shall neglect this weak dependence. In order to add to the understanding of the nature of these coefficients, as well as of the approximation made in the derivation of equation (104), a particular problem is solved explicitly in Appendix B.

where $\bar{\chi}_N = \chi_N(\eta = \infty)$. The energy $W(Q)$ in this case depends on β only. The equilibrium deformation β_0 always differs from zero, except in the case of the completely closed shell. Therefore, the spherical shape turns out to be unstable for any number of outside nucleons and the deformation increases smoothly when the occupation of the shell increases.

For a fixed value of η, the equilibrium deformation β_0 determined from (106) is equal to

$$\beta_0(\eta) = \frac{q_0}{(1-\varkappa/k)\,\overline{Q}}\,\Lambda(\eta). \tag{108}$$

This equation can be written as

$$Q_\lambda = (1-\varkappa/k)\,Q, \tag{108'}$$

which indicates that the quantity $\varkappa/k$ describes the polarizability of the core by the outside nucleons*.

In the case of equilibrium between β and η, $W(Q)$ takes the form of

$$W(Q) = W_0 - \frac{\varkappa q_0^2}{2(1-\varkappa/k)}\,\Lambda^2(\eta) - \frac{\Delta^2}{G}. \tag{109}$$

This equation is seen to be equivalent to (96), (98). The only effect of the deformable core ($k \neq \infty$) is an effective increase in the quadrupole force $\left(\varkappa_{\mathrm{eff}} = \dfrac{\varkappa}{1-\varkappa/k}\right)$.

For small η, one finds from (57) and (100)

$$\left.\begin{aligned} \frac{\Delta^2}{G} &\approx \frac{\Omega^2}{4}\,G\Theta_N - \frac{\Omega^2}{12}\,G\Theta_N\left(1-\frac{\xi^2}{3}\right)\eta^2 \\ \Lambda(\eta) &\approx \frac{\Omega}{6}\left(1-\frac{\xi^2}{3}\right)\Theta_N\,\eta \end{aligned}\quad (\eta < 1).\right\} \tag{110}$$

Combining (109) and (110) we get

$$W(Q) = W_0 - \frac{\Omega^2}{4}\,G\Theta_N + \frac{1}{2}\,C_\eta\,\eta^2. \tag{111}$$

where

$$C_\eta = \frac{\Omega^2}{6}\,G\Theta_N\left(1-\frac{\xi^2}{3}\right)\left[1-\frac{\varkappa q_0^2}{6\,G(1-\varkappa/k)}\left(1-\frac{\xi^2}{3}\right)\Theta_N\right]. \tag{112}$$

* The quantity $\varkappa/k$ can be estimated empirically from the values of the quadrupole moment for nuclei with one particle outside of closed shells (e. g., O^{17} or Bi^{209}). These nuclei exhibit quadrupole moments of the order of single-particle values, which implies that $\varkappa/k \sim 0.5$.

The stability of the spherical shape depends on the ratio of the two terms in the square brackets of (112). The first term, which is associated with the pairing interaction, tends to produce stability. The second term gives the effect of the quadrupole interaction between nucleons and, in the case of attraction ($\varkappa > 0$), tends to produce instability of the spherical shape. Introducing the quantity Θ_{N_0} by the equation

$$\frac{1}{\Theta_{N_0}} = \frac{\varkappa q_0^2}{6\,G\,(1-\varkappa/k)}\left(1-\frac{\xi^2}{3}\right), \tag{113}$$

we can rewrite (112) in the form

$$C_\eta = \frac{\Omega^2}{6}\,G\,\Theta_N\left(1-\frac{\xi^2}{3}\right)(1-\Theta_N/\Theta_{N_0}). \tag{114}$$

The quantity Θ_{N_0} represents the value of the occupation of the unfilled shell required to make the spherical shape unstable. The value of Θ_{N_0} changes from shell to shell. If $\Theta_{N_0} > 1$, then the nucleus remains spherical for any occupation of this shell.

When the condition $\Theta_{N_0} < \Theta_N \leqslant 1$ is fulfilled, the spherical nucleus is unstable. The equilibrium deformation is determined in this case by the extremum of (109) for $\eta \neq 0$. To simplify the calculations, we restrict ourselves to the case of uniform level density ($\xi = 0$). Using (57), (100), and (113), one finds from (109)

$$W(Q) = W_0 - \frac{\Omega^2}{4}\,G\,\Theta_N\left(\frac{\eta^2}{\sinh\eta} + \frac{3}{4}\,\Theta_N/\Theta_{N_0}\cdot\gamma^2\right). \tag{115}$$

The extremum of (115) is determined by the equation

$$\left(\coth\eta_0 - \frac{1}{\eta_0}\right)\left[1-\frac{3}{2}\,\Theta_N/\Theta_{N_0}\cdot\frac{\gamma(\eta_0)}{\eta_0}\right] = 0. \tag{116}$$

The solution corresponding to the first factor in (116) gives an extremum for $\eta_0 = 0$. Since $3\gamma/2\eta < 1$, the second solution of (116) occurs only for $\Theta_N > \Theta_{N_0}$. Using this solution, one finds the equilibrium deformation β_0 from (108):

$$\beta_0 = \frac{\Omega\,q_0\,\Theta_N}{4\,(1-\varkappa/k)\,\overline{Q}}\,\gamma_0'. \tag{117}$$

In the absence of pairing interaction, the equilibrium deformation is given by the same equation (117) without the factor γ_0. Therefore, this factor represents a relative reduction of the equilibrium deformation arising from the

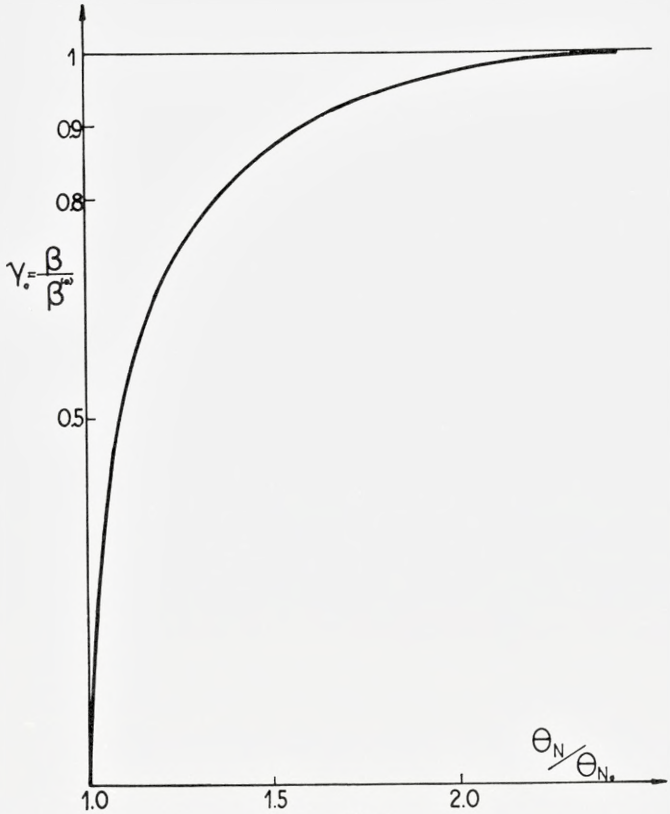

Fig. 1. Relative reduction of the equilibrium deformation arising from the pairing interaction.
The figure shows the ratio of the equilibrium deformation to that in the absence of pairing interaction, in the region of deformed nuclei ($\Theta_N < \Theta_{N_0}$). The reduction factor γ_0 is obtained from the equation (116).

pairing interaction. The function $\gamma_0\,(\Theta_N/\Theta_{N_0})$, determined from the equation (116), rises rapidly with increasing occupation Θ_N near Θ_{N_0} and quickly approaches its limiting value $\gamma_0 = 1$. (see Fig. 1). Near the point of instablity, one finds

$$\gamma_0 \approx \frac{2\,\Theta_{N_0}}{3\,\Theta_N}\sqrt{\frac{15}{2}\left(1-\frac{\Theta_{N_0}}{\Theta_N}\right)}; \quad \left(\eta_0 = \sqrt{\frac{15}{2}\left(1-\frac{\Theta_{N_0}}{\Theta_N}\right)} \lesssim 1\right) \tag{118}$$

and from (117) it follows that

$$\beta_0 \approx \frac{q_0\,\Omega\Theta_{N_0}}{6\,(1-\varkappa/k)\,\overline{Q}}\sqrt{\frac{15}{2}\left(1-\frac{\Theta_{N_0}}{\Theta_N}\right)}. \tag{119}$$

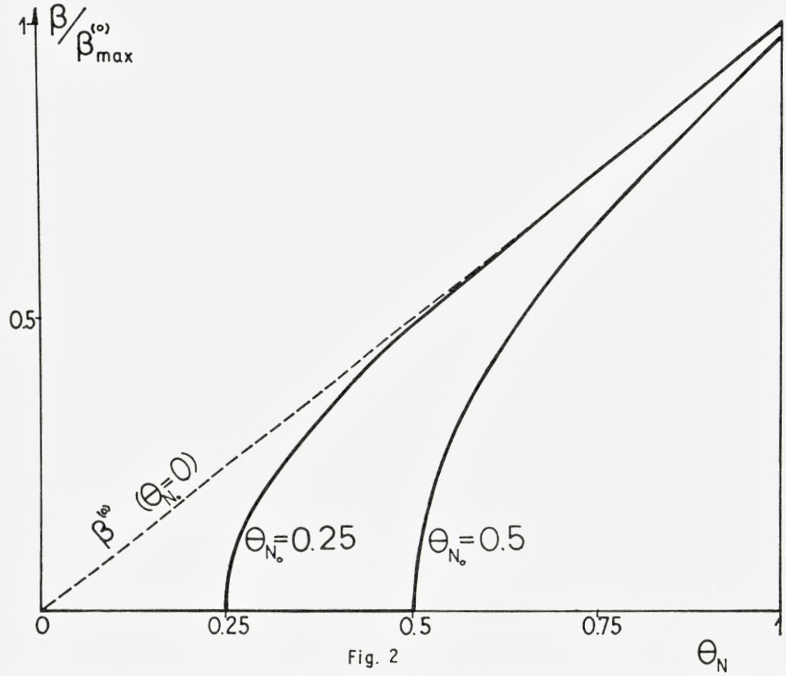

Fig. 2

Fig. 2. Dependence of the equilibrium deformation on the occupation of the unfilled shell. The equilibrium deformation β_0 is plotted as a function of the occupation factor Θ_N for two different values of the quantity Θ_{N_0} given by (113).
The dashed line shows the deformation $\beta^{(0)}$ in the absence of pairing interaction. The maximum value of $\beta^{(0)}$ (for a half-filled shell, i.e., $\Theta_N = 1$) is chosen as a unit.

Therefore, the transition from the spherical nucleus to the deformed one is rather sharp. The minimum value of the possible deformation may be estimated from (119) by setting $N = N_0 + 1$ (i.e. $(1 - \Theta_{N_0}/\Theta_N) \approx N_0^{-1}$):

$$\beta_{\min} \approx \frac{q_0}{3\,(1-\varkappa/k)\,\overline{Q}} \sqrt{\frac{15}{2} N_0}. \tag{120}$$

For large values of Θ_N/Θ_{N_0}, one finds from (116)

$$\gamma_0 \approx 1 - 2\,(3\,\Theta_N/\Theta_{N_0} - 1)\exp(-3\,\Theta_N/\Theta_{N_0}) \tag{121}$$

and, therefore,

$$\beta_0 \approx \frac{q_0\,\Omega\Theta_N}{6\,(1-\varkappa/k)\,\overline{Q}} \left\{1 - 2\,(3\,\Theta_N/\Theta_{N_0} - 1)\exp(-3\,\Theta_N/\Theta_{N_0})\right\}. \tag{122}$$

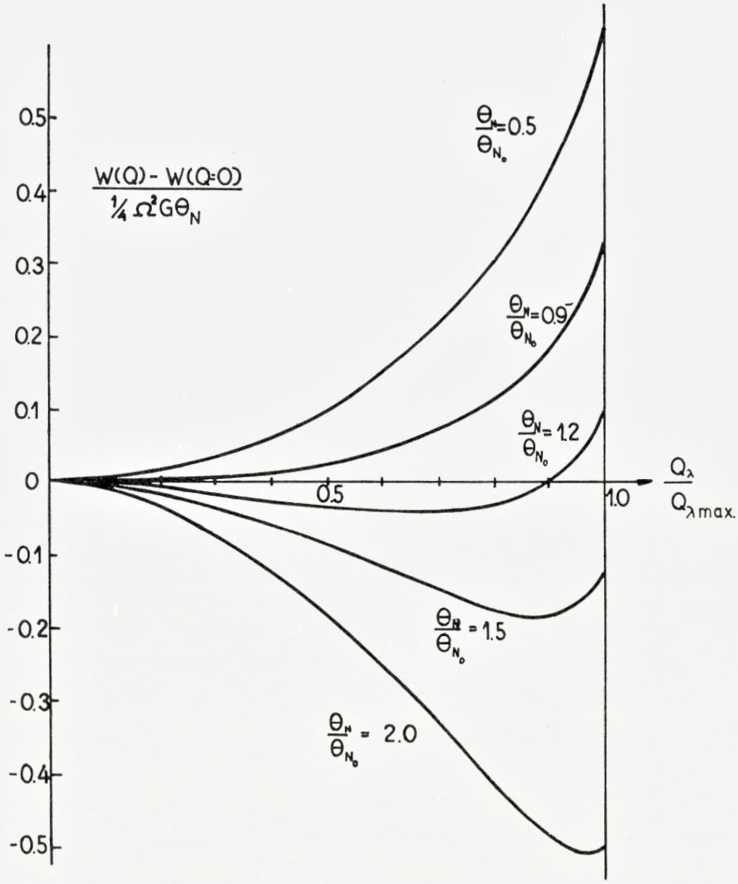

Fig.3

Fig. 3. Dependence of the nuclear energy on the configuration of the particles in the unfilled shell.

The figure shows the nuclear energy as a function of the quadrupole moment of outside nucleons for five different occupation factors. The energy is measured as the difference from that in a spherical nucleus, in units $\frac{1}{4}\Omega^2 G\Theta_N$ (total pairing energy in the spherical nucleus). The quadrupole moment, Q_λ, is plotted in units of its maximum value $Q_{\lambda\,\max} = q_0 \frac{\Omega}{4}\Theta_N$. (The weak dependence of Q_λ on the change in wave functions upon deformation has been neglected).

In this region (near the middle of the shell), the main dependence β_0 of the occupation is given in (122) by the quantity Θ_N; thus, we may write approximately

$$\beta_0 \approx \beta_{\max} \Theta_N, \tag{123}$$

where $\beta_{\max}$ is the value of the deformation for the maximum occupation of the shell ($\Theta_N = 1$):

$$\beta_{\max} \approx \frac{q_0 \Omega}{6(1 - \varkappa/k)\,\overline{Q}}. \tag{124}$$

Consequently, the values of the equilibrium deformation have a lower bound and change smoothly near the middle of the shell.

The equilibrium deformation β_0 as a function of the occupation factor Θ_N is illustrated in Fig. 2, where the dashed line corresponds to the absence of pairing interaction.

It is of interest to point out the dependence of $\beta_{\min}$ and $\beta_{\max}$ on the atomic number A. Since the quantity $\varkappa \sim A^{-7/3}$ (cf. footnote, p. 29) and $G \sim A^{-1}$*), $q_0 \sim R_0^2 \sim A^{2/3}$, it follows from (113) that Θ_{N_0} is independent of A (and, therefore, $N_0 \sim \Omega \sim A^{2/3}$). From (105) we have also $\overline{Q} \sim A^{5/3}$. Using these facts, we obtain from (120) and (124)

$$\beta_{\min} \sim A^{-2/3}; \quad \beta_{\max} \sim A^{-1/3}, \tag{125}$$

which is in agreement with the observed trends**.

The dependence of the nuclear energy on the configuration of the outside particles (for the equilibrium deformation of the core), which is given by the equation (115), is illustrated in Fig. 3. The value of the quadrupole moment of the outside nucleons is chosen as the variable. Due to the finite nature of the quadrupole moment Q_λ, the curves have terminal points. This fact may violate the possibility for vibrations of outside nucleons in a deformed field.

3. Inertial Parameter

As is seen from (89), for the calculation of the inertial parameter $B(\vartheta)$ it is necessary to know the dependence of the ground-state wave function on the collective parameter ϑ. To this end, we consider ψ_0 in the representation of occupation numbers of the old particles (78)

$$\Psi_0 = \prod_\nu (U_\nu + V_\nu b_{\nu+}^+ b_-^+)\Psi_0^{(0)}. \tag{126}$$

* Here, we neglect effects of intershell transitions, which changes this dependence to some degree (cf. p. 17).

** For strongly deformed nuclei, the levels of different shells cross[11]. Redistribution of the particles might occur, however, only when the levels of the low shell cross the empty levels of the unfilled shell. The number of such crossings is rather small, and we may expect that they do not change qualitatively the results concerning the equilibrium deformation.

The change of the parameter ϑ means a certain variation of the self-consistent field; in the rotational case, it is equivalent to a certain rotation; in the case of quadrupole vibrations of a spherical nucleus, it is a quadrupole deformation, and so on. In the general case, we can associate with ϑ a certain operator of the infinitesimal displacement K^{ϑ}. The associated variation of the self-consistent field changes the single-particle states and, therefore, the operators of the old particles $b^+_{\nu\sigma}$. Since the $b^+_{\nu\sigma}$ correspond to independent particles, the operator K^{ϑ} expressed in terms of $b^+_{\nu\sigma}$ may be represented by a sum of single-particle operators

$$K^{\vartheta} = \sum_{\nu\nu'} k^{\vartheta}_{\nu\nu'} \left(b^+_{\nu+} b_{\nu'+} \pm b^+_{\nu'-} b_{\nu-}\right), \tag{127}$$

where $k^{\vartheta}_{\nu\nu'} = \langle \nu + \mid k^{\vartheta} \mid \nu' + \rangle$ is the matrix element of the single-particle operator. The sign $\pm$ is due to the condition

$$\langle \nu' - \mid k^{\vartheta} \mid \nu - \rangle = \pm \langle \nu + \mid k^{\vartheta} \mid \nu' + \rangle$$

and is defined by the behaviour of the operator k under time reversal.

In addition to the change in the wave functions of the original particles, the deformation causes a shift of the single-particle energy levels, which gives rise to a change in U_ν and V_ν. Therefore, the total effect of the deformation can be written in the following form:

$$\frac{\partial}{\partial\vartheta} = -i K^{\vartheta} + \left(\frac{\partial}{\partial\vartheta}\right)_b, \tag{128}$$

where the last derivative is taken, keeping the operators $b_{\nu\sigma}$ constant. Performing in (127) the transition to the operators α, β we get

$$\left.\begin{aligned} K^{\vartheta} = \sum_{\nu} k^{\vartheta}_{\nu\nu} (1 \pm 1) V^2_\nu + \sum_{\nu\nu'} k^{\vartheta}_{\nu\nu'} (U_\nu U_{\nu'} \mp V_\nu V_{\nu'}) (\alpha^+_\nu \alpha_{\nu'} \pm \beta^+_{\nu'} \beta_\nu) \\ + \sum_{\nu\nu'} k^{\vartheta}_{\nu\nu'} (U_\nu V_{\nu'} \pm V_\nu U_{\nu'}) (\alpha^+_\nu \beta^+_{\nu'} \pm \beta_\nu \alpha_{\nu'}). \end{aligned}\right\} \tag{129}$$

The result of the operation of K^{ϑ} on the vacuum state is given by

$$K^{\vartheta} \Psi_0 = \sum_{\nu} k^{\vartheta}_{\nu\nu} (1 \pm 1) V^2_\nu \cdot \Psi_0 + \sum_{\nu\nu'} k^{\vartheta}_{\nu\nu'} (U_\nu V_{\nu'} \pm V_\nu U_{\nu'}) \alpha^+_\nu \beta^+_\nu \Psi_0. \tag{130}$$

Now, consider the last operator in (128). From (126) it follows that

$$\left(\frac{\partial \Psi_0}{\partial\vartheta}\right)_b = \sum_{\nu} \left(\frac{\partial U_\nu}{\partial\vartheta} + \frac{\partial V_\nu}{\partial\vartheta} b^+_{\nu+} b^+_{\nu-}\right) \prod_{\nu' \neq \nu} (U_{\nu'} + V_{\nu'} b^+_{\nu'+} b^+_{\nu'-}) \Psi^{(0)}_0. \tag{131}$$

With the aid of the normalization condition $U^2 + V^2 = 1$, the first factor inside the sum takes the form

$$\frac{\partial U_\nu}{\partial \vartheta} + \frac{\partial V_\nu}{\partial \vartheta} b_{\nu+}^+ b_{\nu-}^+ = -\frac{1}{V_\nu}\frac{\partial U_\nu}{\partial \vartheta}(U_\nu b_{\nu+}^+ b_{\nu-}^+ - V_\nu). \tag{132}$$

Thus, by comparing (131) with (80), we get

$$\left(\frac{\partial \Psi_0}{\partial \vartheta}\right)_b = -\sum_\nu \frac{1}{V_\nu}\frac{\partial U_\nu}{\partial \vartheta}\alpha_\nu^+ \beta_\nu^+ \Psi_0. \tag{133}$$

As it is seen from (130) and (133), $\frac{\partial}{\partial \vartheta}$ causes transitions from the vacuum only to states with two quasi-particles

$$\langle \alpha_\nu^+ \beta_{\nu'}^+ \left| \frac{\partial}{\partial \vartheta} \right| 0 \rangle = -ik_{\nu\nu'}^\vartheta (U_\nu V_{\nu'} \pm V_\nu U_{\nu'}) - \frac{\delta_{\nu\nu'}}{V_\nu}\frac{\partial U_\nu}{\partial \vartheta}. \tag{134}$$

There is no interference between two terms in (134) provided the diagonal matrix element $k_{\nu\nu}^\vartheta$ is equal to zero. In this case, the inertial parameter for the vacuum state is given by

$$\begin{aligned} B(\vartheta) &\equiv B_1 + B_2 \\ &= 2\hbar^2 \sum_{\nu\nu'} \frac{|k_{\nu\nu'}^\vartheta|^2 (U_\nu V_{\nu'} \pm V_\nu U_{\nu'})^2}{E_\nu + E_{\nu'}} + \hbar^2 \sum_\nu \frac{1}{V_\nu^2}\left(\frac{\partial U_\nu}{\partial \vartheta}\right)^2 \frac{1}{E_\nu}, \end{aligned} \tag{135}$$

where E_ν is the energy of a quasi-particle (71). The expression (135) defines the inertial parameter for even-even nuclei. The ground state of odd nuclei is given by the function of the form (79), say, $\alpha_{\nu_0}^+ \psi_0$. Performing similar calculations with this function we get

$$\left.\begin{aligned} B_{\mathrm{odd}} &= 2\hbar^2 \sum_{\substack{\nu\nu' \\ \nu' \neq \nu_0}}{}' \frac{|k_{\nu\nu'}^\vartheta|^2 (U_\nu V_{\nu'} \pm V_\nu U_{\nu'})^2}{E_\nu + E_{\nu'}} \\ &+ \hbar^2 \sum_{\nu \neq \nu_0} \left(\frac{1}{V_\nu}\frac{\partial U_\nu}{\partial \vartheta}\right)^2 \frac{1}{E_\nu} + 2\hbar^2 \sum_{\nu \neq \nu_0} \frac{|k_{\nu\nu_0}^\vartheta|^2 (U_\nu U_{\nu_0} \mp V_\nu V_{\nu_0})}{E_\nu - E_{\nu_0}}. \end{aligned}\right\} \tag{136}$$

This expression can be rewritten in the form

$$B_{\mathrm{odd}} = B_1 + B_2 + B_3, \tag{136'}$$

where B_1 and B_2 are given by (135) and correspond to even-even nuclei.

The term B_3, which determines the difference in the inertial parameters for neighbouring even-even and odd-A nuclei, is then given by

$$\left.\begin{aligned} B_3 = 2\hbar^2 \sum_{\nu \neq \nu_0} \frac{|k^{\vartheta}_{\nu\nu_0}|^2}{E_\nu^2 - E_{\nu_0}^2} \{[(U_\nu^2 - V_\nu^2)(U_{\nu_0}^2 - V_{\nu_0}^2) \pm 4\,U_\nu V_\nu U_{\nu_0} V_{\nu_0}]\,E_\nu + E_{\nu_0}\} \\ -\frac{\hbar^2}{E_{\nu_0}}\left(\frac{1}{V_{\nu_0}}\frac{\partial U_{\nu_0}}{\partial\vartheta}\right)^2. \end{aligned}\right\} \qquad (136'')$$

The two terms in (135) have an essentially different nature. In terms of original particles a nuclear deformation, which corresponds to the collective parameter ϑ, gives rise, in the first place, to single-particle transitions into the higher states and, secondly, to a change of the Fermi sea without particle excitations. B_1 corresponds to the first effect. The pairing interaction does not change the structure of this term, but makes only quantitative alterations, which are connected with the new energy spectrum and the new distribution of the particles ($U_\nu, V_\nu \neq 0{,}1$). The term B_2 is connected with the change of the Fermi sea by the deformation. This qualitatively new effect is caused by the pairing interaction and disappears for non-interacting particles (when U_ν, V_ν = const). In the case of rotations, the term B_2 is equal to zero, since U_ν, V_ν do not depend on the nuclear orientation. For vibrations, on the other hand, the term B_2 will be shown to make the main contribution in the inertial parameter. Therefore, the pairing interaction basically changes the character of the vibrational motion.

4. Rotational Moment of Inertia

In the case of rotations of axially symmetric nuclei about an axis perpendicular to the nuclear symmetry axis, the mass parameter B gives the moment of inertia J. In this case, k is the operator of the particle angular momentum j_x. Therefore, one finds from (135) for the moment of inertia of even-even nuclei

$$J = 2\hbar^2 \sum_{\nu\nu'} \frac{|(j_x)_{\nu\nu'}|^2}{E_\nu + E_{\nu'}} (U_\nu V_{\nu'} - V_\nu U_{\nu'})^2. \qquad (137)$$

Since j_x changes sign under time reversal, the sign $(-)$ has been chosen in (135).

In the absence of the pairing interaction, we have U_ν, $V_\nu = 1{,}0$ and $E_\nu = E_\nu^0 = |\varepsilon_\nu - \lambda|$, and (137) takes the form

$$J^{(0)} = 4\hbar^2 \sum_{\nu\nu'} \frac{|(j_x)_{\nu\nu'}|^2}{E_\nu^0 + E_{\nu'}} U_\nu^2 V_{\nu'}^2 = 4\hbar^2 \sum_{\varepsilon_\nu > \lambda > \varepsilon_{\nu'}} \frac{|(j_x)_{\nu\nu'}|^2}{\varepsilon_\nu - \varepsilon_{\nu'}}, \qquad (138)$$

which coincides with the ordinary value for the moment of inertia in an independent-particle model.

Inserting the expressions (26) for U_ν, V_ν in (137), we get for the case with a constant gap Δ:

$$J = \hbar^2 \sum_{\nu\nu'} \frac{|(j_x)_{\nu\nu'}|^2}{E_\nu + E_{\nu'}} \left[1 - \frac{\Delta^2 + (\varepsilon_\nu - \lambda)(\varepsilon_{\nu'} - \lambda)}{E_\nu E_{\nu'}}\right]. \tag{139}$$

In order to analyse this expression we split it into two parts. $J = J' + J''$, where

$$J' = 2\hbar^2 \sum_{\varepsilon_\nu > \lambda > \varepsilon_{\nu'}} \frac{|(j_x)_{\nu\nu'}|^2}{E_\nu + E_{\nu'}} \left(1 + \frac{E_\nu^0 E_{\nu'}^0 - \Delta^2}{E_\nu E_{\nu'}}\right), \tag{140}$$

$$J'' = \hbar^2 \Delta^2 \sum_{(\varepsilon_\nu - \lambda)(\varepsilon_{\nu'} - \lambda) > 0} \frac{|(j_x)_{\nu\nu'}|^2}{E_\nu + E_{\nu'}} \quad \frac{(\varepsilon_\nu - \varepsilon_{\nu'})^2}{E_\nu E_{\nu'} (E_\nu E_{\nu'} + E_\nu^0 E_{\nu'}^0 + \Delta^2)}. \tag{141}$$

J' contains only transitions connecting states below with those above the Fermi surface. Comparing (138) and (140) one can see that $J' \leqslant J^{(0)}$ (an equality is achieved for non-interacting particles, when $\Delta = 0$). A decrease of J' with respect to $J^{(0)}$ occurs, in the first place, because of an increase of the energy denominator and, secondly, owing to the modification of the Fermi sea. (The probability of finding an occupied state below the Fermi surface and an empty state above has decreased). The term J'' contains transitions only on one side of the Fermi surface and gives a relatively small contribution for strongly deformed nuclei.

To estimate the term J' we rewrite (140) in the form

$$J' = 4\hbar^2 \sum_{\varepsilon_\nu > \lambda > \varepsilon_{\nu'}} \frac{|(j_x)_{\nu\nu'}|^2}{E_\nu^0 + E_{\nu'}^0} \left[\frac{E_\nu^0 + E_{\nu'}^0}{E_\nu + E_{\nu'}^0} \cdot \frac{1}{2}\left(1 + \frac{E_\nu^0 E_{\nu'}^0 - \Delta^2}{E_\nu E_{\nu'}}\right)\right]. \tag{142}$$

Assuming that $(j_x)_{\nu\nu'}$ is a sharp function of ν and ν', we may take out of the sum the smoothly changing factor in the square brackets, evaluating it for certain average values E_ν^0 and $E_{\nu'}^0$. Assuming that $\bar{E}_\nu^0 \approx \bar{E}_{\nu'}^0 = E^0$, we obtain

$$J'/J_{\text{rig}} = \left(1 + \frac{\Delta}{E^0}\right)^{2-3/2}, \tag{143}$$

where J_{rig} is the moment of inertia in the absence of the interaction ($\Delta \to 0$) which coincides with the moment for rigid rotations. It is seen that the moment of inertia is rather sensitive to the effect of the pairing correlations;

thus, for $\Delta \approx (0.7-0.8)\,E^0$, which approximately corresponds to the situation in the most strongly deformed nuclei, one obtains $J'/J_{\text{rig}} \approx \frac{1}{2}$.

In order to estimate the term J'', we assume that the matrix element $j_{\nu\nu'}$ connects the levels, separated by the same energy as in (143), namely, $\varepsilon_{\nu'} - \varepsilon_\nu \approx 2\,E^0$. Then, introducing a certain average distance of such two levels from the Fermi surface,

$$E^0_{\bar\nu} \approx \frac{\varepsilon_\nu + \varepsilon_{\nu'}}{2} - \lambda$$

and using the approximate relations

$$\begin{aligned} E_\nu E_{\nu'} &\approx E^2_{\bar\nu} = E^{02}_{\bar\nu} + \Delta^2 \\ E_\nu + E_{\nu'} &\approx 2\,E_{\bar\nu} = 2\sqrt{E^{02}_{\bar\nu} + \Delta^2}, \end{aligned}$$

we obtain from (141)

$$J'' = \hbar^2 \Delta^2 \sum_{\lambda<\varepsilon_{\nu'}<\varepsilon_\nu}{}' \frac{|j_{\nu\nu'}|^2}{\varepsilon_\nu - \varepsilon_{\nu'}} \quad \frac{(\varepsilon_\nu - \varepsilon_{\nu'})^3}{(E^{02}_{\bar\nu} + \Delta^2)^{5/2}}.$$

Assuming, further, that the value of $|j_{\nu\nu'}|^2$ does not strongly vary in the effective region of the sum, and taking into account that, in this case,

$$4\,\hbar^2 \frac{|j_{\nu\nu'}|^2}{\varepsilon_\nu - \varepsilon_{\nu'}} \approx J^{(0)}$$

we find

$$J''/J^{(0)} \approx \frac{1}{4} \Delta^2 (2\,E^0)^3 \sum_{\bar\nu > \bar\nu_F}{}' \frac{1}{(E^{02}_{\bar\nu} + \Delta^2)^{5/2}}. \tag{144}$$

For simplicity, we consider an oscillator-like level scheme, where $E^0_{\bar\nu} = 2\,E^0 \bar\nu$ $(\bar\nu = 1,2\ldots)$. For strongly deformed nuclei, when $\frac{\Delta}{2\,E^0} < 1$, the terms in the sum (144) decrease very rapidly, and we may restrict ourselves only to the first term. Then, we obtain

$$J''/J^{(0)} \approx \frac{1}{16}\left(\frac{\Delta}{E^0}\right)^2; \quad \left(\frac{\Delta}{2\,E^0} < 1\right),$$

which is very small compared to the ratio (143). When we go to less deformed nuclei, the ratio $\frac{\Delta}{2\,E^0}$ increases and more transitions make a significant contribution to the moment of inertia. In the case of $\frac{\Delta}{2\,E^0} \gg 1$, all transitions inside the energy region Δ are almost equivalent, but only one of them is included in J' (viz. that which crosses the Fermi surface). Therefore, we can expect that, in this case, the J''-term becomes larger than J' by the factor $\sim \frac{\Delta}{E^0}$ (which represents the number of effective transitions). For $\frac{\Delta}{2E^0} > 1$, we can replace the sum in (144) by an integral. Then, we obtain

$$J''/J^{(0)} \approx \frac{2}{3}\left(\frac{E^0}{\Delta}\right)^2; \quad \left(\frac{\Delta}{2E^0} > 1\right),$$

which confirms our expectation.

5. Inertial Parameter for Quadrupole Vibrations of Spherical Nuclei

As has been shown above, deviations of nuclei from the spherical equilibrium shape can be characterized by two parameters: the quadrupole moment of the closed-shell core Q_{cl} and that of the outside nucleons Q_λ (or by the parameter η proportional to Q_λ). The deformation associated with Q_{cl} changes only single-particle states, so that it contributes only to the term B_1 in (135). In the harmonic oscillator model, the operator K in this term is proportional to the single-particle quadrupole moment. Since the quadrupole transitions inside one shell are forbidden, the value B_1 in this case is very small, because the energy denominator is large. Let us introduce the deformation β' of the closed-shell core connected with Q_{cl} by the equation (105). Then the inertial parameter B_β, related to β' in the absence of pairing interaction coincides with that for the oscillations of an irrotational liquid drop[19)]

$$B_{\text{irr}} = \frac{3}{8\pi} AmR_0^2 \tag{145}$$

(m is the nucleon mass, R_0 and A are the radius and the atomic number of the nucleus). One might expect that this result is not sensitive to the model. The pairing interaction does not significantly change the value of $B_{\beta'}$.

The inertial coefficient B_η related to η is given only by the term B_2 in (135)

$$B_\eta = \hbar^2 \sum_\nu{}' \left(\frac{1}{V_\nu}\frac{\partial U_\nu}{\partial \eta}\right)^2 \frac{1}{E_\nu}. \tag{146}$$

The parameter B_η is absent for non-interacting particles and, therefore, is of special interest.

Using (26) and (28) we obtain

$$\frac{1}{V_\nu}\frac{\partial U_\nu}{\partial \eta} = \frac{1}{2\,U_\nu V_\nu}\frac{\partial U_\nu^2}{\partial \eta} = \frac{1}{2\,E_\nu^2}\left[\varDelta\frac{\partial}{\partial \eta}(\varepsilon_\nu - \lambda) - (\varepsilon_\nu - \lambda)\frac{\partial \varDelta}{\partial \eta}\right]. \tag{147}$$

In the case of vibrations of spherical nuclei, we are interested in the value B_η for $\eta = 0$. The expansion of $\varDelta$ for small η, as can be seen from (110), does not contain a linear term and therefore $\frac{\partial \varDelta}{\partial \eta}$ vanishes for $\eta = 0$. The quantity $\varepsilon_\nu - \lambda$ for small η is given by

$$\varepsilon_\nu - \lambda \approx \frac{1}{2}\,\Omega G \chi_N^0 + \Omega G \frac{\varepsilon_\nu - \varepsilon_\lambda}{\varepsilon'' - \varepsilon'}\,\eta. \tag{148}$$

The average energy of the λ-shell, in the first approximation, is not shifted by the deformation. The ratio $(\varepsilon_\nu - \varepsilon_\lambda)/(\varepsilon'' - \varepsilon')$ remains constant for small η and, therefore,

$$\frac{\partial}{\partial \eta}(\varepsilon_\nu - \lambda) \approx \Omega G \frac{\varepsilon_\nu - \varepsilon_\lambda}{\varepsilon'' - \varepsilon'}. \tag{149}$$

With the aid of (149) and (147) we get from (146)

$$B_\eta = \frac{1}{4}\hbar^2 \sum_\nu \frac{\varDelta^2}{E_\nu^5}(\Omega G)^2 \left(\frac{\varepsilon_\nu - \varepsilon_\lambda}{\varepsilon'' - \varepsilon'}\right)^2. \tag{150}$$

Inserting in (150) the values of $\varDelta$ and E_ν, which for $\eta \ll 1$ are given by

$$\varDelta \approx \frac{1}{2}\,\Omega G \Theta_N^{1/2}; \qquad E_\nu \approx \frac{1}{2}\,\Omega G,$$

we get

$$B_\eta = \frac{2\,\hbar^2 \Theta N}{\Omega G} \sum_\nu \left(\frac{\varepsilon_\nu - \varepsilon_\lambda}{\varepsilon'' - \varepsilon'}\right)^2. \tag{151}$$

Replacing the sum in (151) by an integral, we obtain

$$B_\eta = \frac{\hbar^2}{6\,G}\left(1 - \frac{\xi^2}{3}\right)\Theta_N. \tag{152}$$

The equation (152) contains the value of the interaction G in the denominator. It must be pointed out that the transition in (152) to the limit $G \to 0$ is not valid, since the condition $\eta \approx (\varepsilon'' - \varepsilon')/\Omega G < 1$ has been used in its calculation. As can be seen from the exact formula (146), $B_\eta \to 0$ for

$G \to 0$. When G decreases, then the number of outside particles, needed to make the spherical shape unstable, also descreases. For a certain value of G, the spherical shape becomes unstable even for one outside pair ($N = 2$), which makes the inertial parameter (152) meaningless. The minimum value of G in the equation (152) can be estimated from (113) if one requires that $\Theta_{N_0} > \Theta_2$, which leads to

$$G > \frac{2}{3} \frac{1 - \frac{\xi^2}{3}}{1 - \varkappa/k} \frac{\varkappa q_0^2}{\Omega} \approx \frac{\varkappa q_0^2}{\Omega}. \tag{153}$$

To analyze the quantity B_η we compare it with the mass coefficient B_{irr} (145). To this purpose, we need a relation between η and the equivalent deformation of the nucleus in the hydrodynamical model. Assuming the equilibrium ratio between β and η we get from (108) and (110)*

$$\left(\frac{\eta}{\beta}\right)_{eq} = \frac{6\,\bar{Q}\,(1 - \varkappa/k)}{q_0 \left(1 - \frac{\xi^2}{3}\right) \Omega \Theta_N}. \tag{154}$$

From (145), (152), and (154) it follows that

$$\left(\frac{\eta}{\beta}\right)^2 \frac{B_\eta}{B_{irr}} \approx \frac{16\,\pi\,(1 - \varkappa/k)^2}{1 - \frac{\xi^2}{3}} \cdot \frac{\hbar^2}{m R_0^2} \left(\frac{\bar{Q}}{q_0 A}\right)^2 \frac{A}{\Omega^2 G \Theta_N}. \tag{155}$$

Using (105), and setting $q_0/R_0^2 = 2$; $\xi = 1$ according to the oscillator model, one obtains from (155)

$$\left(\frac{\eta}{\beta}\right)^2 \frac{B_\eta}{B_{irr}} \approx 100 \text{ MeV } (1 - \varkappa/k)^2 \frac{A^{1/3}}{\Omega G N}. \tag{156}$$

The ratio (156) is significantly larger than unity and is in qualitative agreement with the observed trends. Thus, for the single particle, excitation energy $\Omega G \approx 1.5$ MeV and $\varkappa/k = 0.5$, $A^{1/3} = 5$; $N = 6$ the ratio is equal to 10 and decreases when the number of outside particles increases.

6. Normal Vibrations of Spherical Nuclei

The potential energy of collective vibrations is given by (104). Introducing the variables η and β' (the deformation of the core, associated with Q_{cl} by: $Q_{cl} = \bar{Q}\beta'$), we obtain for small η

* In the general case, the equilibrium polarizability of the core $\varkappa/k$ may be replaced by $\alpha\varkappa/k$, where $0 \leqslant \alpha \leqslant 1$.

$$\left.\begin{aligned} W(Q) = W_0 - \frac{\Omega^2}{4} G\Theta_N + \frac{1}{2}(k-\varkappa)\bar{Q}^2\beta'^2 - \frac{\Omega}{6}\varkappa q_0 \bar{Q}\left(1-\frac{\xi^2}{3}\right)\Theta_N \beta'\eta \\ + \frac{\Omega^2}{12} G\Theta_N\left(1-\frac{\xi^2}{3}\right)\left[1-\frac{\varkappa q_0^2}{6\,G}\left(1-\frac{\xi^2}{3}\right)\Theta_N\right]\eta^2. \end{aligned}\right\} \quad (157)$$

As has been established in the previous section, the kinetic energy has the form

$$T = \frac{1}{2}B_\eta \dot{\eta}^2 + \frac{1}{2}B_{\beta'}\dot{\beta}'^2, \quad (158)$$

where B_η is given by (152). Because of the smallness of the coefficient $B_{\beta'}$, the second term in (158) is much smaller than the first one. Making use of this fact in the transformations of (157) and (158) to the normal vibrations, we obtain for the normal coordinates*

$$\alpha_1 \approx \sqrt{B_\eta}\left\{\eta + \frac{\varkappa q_0 \Omega\Theta_N}{6\,(k-\varkappa)\,\bar{Q}}\left(1-\frac{\xi^2}{3}\right)\frac{B_{\beta'}}{B_\eta}\beta'\right\}, \quad (159)$$

$$\alpha_2 \approx \sqrt{B_{\beta'}}\left\{\beta' - \frac{\varkappa q_0 \Omega\Theta_N}{6\,(k-\varkappa)\,\bar{Q}}\left(1-\frac{\xi^2}{3}\right)\eta\right\}. \quad (160)$$

The corresponding eigenfrequencies are given by

$$\omega_1 = \sqrt{\frac{C_\eta}{B_\eta}}; \qquad \omega_2 = \sqrt{\frac{(k-\varkappa)\,\bar{Q}^2}{B_{\beta'}}}, \quad (161)$$

where C_η is determined by (112) or (114).

The normal vibration of the first type ($\alpha_2 = 0$) preserves the equilibrium relation between β' and η. Indeed, using (98) and (110) and employing the relation between β' and Q_{cl}, one may write the condition $\alpha_2 = 0$ as

$$Q_{cl} = \frac{\varkappa}{k-\varkappa} Q_\lambda \quad \text{or} \quad Q_\lambda = (1-\varkappa/k)\,Q, \quad (162)$$

which is equivalent to the equilibrium relation (108'). Therefore, in the vibration of the first type, the closed-shell core adjusts itself adiabatically to the deformation of the outside nucleons. According to (161), (114), and (152), the energy of this vibration is given by

$$\hbar\omega_1 = \Omega G\sqrt{1-\frac{\Theta_N}{\Theta_{N_0}}} \quad (163)$$

* In terms of these coordinates the Hamiltonian of collective vibrations is given by $H_{coll} = T + W(Q) = \frac{1}{2}(\dot{\alpha}_1^2 + \dot{\alpha}_2^2 + \omega_1^2\alpha_1^2 + \omega_2^2\alpha_2^2)$.

and decreases as the occupation of the shell increases. Only when Θ_N approaches the value Θ_{N_0}, needed to give instability of the spherical shape, does the vibrational energy become appreciably smaller than the intrinsic excitation energy ΩG, as required by the adiabatic condition*. In the absence of pairing interaction this type of vibration vanishes.

The collective motion considered above corresponds to vibrations of the average value of the quadrupole moment. Such a simple physical picture is meaningful only when fluctuations of the quadrupole moment do not exceed the vibrational amplitude. According to (91), the quadratic fluctuation of the qudrupole moment is given by

$$(\delta Q)^2 = \langle \hat{Q}^2 \rangle - \langle \hat{Q} \rangle^2 = \sum_{\nu\nu'} | q_{\nu\nu'} |^2 (U_\nu V_{\nu'} + V_\nu U_{\nu'})^2. \qquad (164)$$

For simplicity, we shall consider only the outside nucleons. In a proper representation, when the states ν are eigenstates in the self-consistent field, the matrix element $q_{\nu\nu'}$ is diagonal (cf. (94)) so that we have

$$(\delta Q)^2 = \sum_{\nu} | q_{\nu\nu} |^2 \, 4 \, U_\nu^2 V_\nu^2.$$

For spherical nuclei ($\eta = 0$), one finds $4 U_\nu^2 V_\nu^2 = \Theta_N$ and, after simple calculations, we obtain

$$(\delta Q_\lambda)^2 = \Theta_N \sum_{\nu} | q_{\nu\nu} |^2 = \frac{\Omega}{12} q_0^2 \Theta_N \left(1 - \frac{\xi^2}{3}\right). \qquad (165)$$

On the other hand, using the relation (98) between Q_λ and η and substituting for η the amplitude of zero vibrations,

$$\eta^0 = \sqrt{\frac{\hbar}{2 B_\eta \omega_1}} = \left[\frac{h\omega_1}{3G} \Theta_N \left(1 - \frac{\xi^2}{3}\right)\right]^{-1/2}, \qquad (166)$$

we find for the zero-vibration amplitude of the quadrupole moment

$$(Q_\lambda^0)^2 = \frac{\Omega}{12} q_0^2 \Theta_N \left(1 - \frac{\xi^2}{3}\right) \frac{\Omega G}{\hbar \omega_1}. \qquad (167)$$

Comparing (165) and (167), we obtain

$$\left(\frac{\delta Q_\lambda}{Q_\lambda^0}\right)^2 = \frac{\hbar\omega_1}{\Omega G} = \sqrt{1 - \frac{\Theta_N}{\Theta_{N_0}}}. \qquad (168)$$

* Here, we are referring to even-even nuclei. In odd nuclei, this adiabatic condition is not fulfilled because of the small excitation energy of the odd particle.

As it is seen from (168), the requirement that this ratio be small coincides with the adiabatic condition.

Consider, now, the second normal vibration ($a_1 = 0$). The closed-shell core participates mainly in this vibration. The outside nucleons are only slightly deformed. The ratio of the amplitudes η and β', in this type of vibrations compared to that in the first type, is given by

$$\left(\frac{\eta}{\beta'}\right)_2 = -\left(\frac{\beta'}{\eta}\right)_1 \frac{B_{\beta'}}{B_\eta} = -\left(\frac{\eta}{\beta'}\right)_1 \left\{\frac{B_{\beta'}}{B_\eta}\left(\frac{\beta'}{\eta}\right)_1^2\right\}. \tag{169}$$

Since $B_{\beta'}$, is of the order of B_{irr} it is seen from (169) that the polarization of the outside nucleons is reduced by the factor $\left(\frac{\beta}{\eta}\right)^2 B_{\mathrm{irr}}/B$ given by (156). The second type of vibration occurs with a high frequency which is determined by the properties of the core and does not depend appreciably on the pairing interaction and on the number of the nucleons in the unfilled shell. Since there is almost no coupling between this vibration and the outside nucleons, the adiabatic condition requires the vibrational energy $h\omega_1$ to be small only compared with the distance between the shells.*

Concluding Remarks

Starting from the basic assumption that a pairing correlation of a "superconducting" type exists between nucleons, we have attempted to investigate consistently the effects of this correlation in different nuclear phenomena.

Although the calculations are based on a rather idealized model, a great number of experimental facts of a different kind are explained in a natural way from a single point of view, viz.,

a) Stability of the spherical shape of nuclei near the closed shells;
b) Sharp transition between spherical and deformed nuclei;
c) Significant reduction of the moment of inertia from the value for rigid rotation;
d) Existence of low-energy vibrations in spherical nuclei near the bound of instability.

* In the oscillator model, $h\omega_2$ turns out to be two times larger than the distance between the shells; this violates the adiabatic condition[18].

The equilibrium deformations, the moment of inertia, the vibrational frequencies, and the inertial parameter obtained from the present model are of the order of magnitude observed, and exhibit a reasonable dependence on the parameters. Besides these collective effects, some particular features of the single-particle spectra are explained (energy gap in even-even nuclei, increased level density just above the gap).

It is outside the scope of this paper to relate the pairing correlation to explicit forms of nucleon-nucleon forces. Here, we are restricting ourselves to a semi-phenomenological description of this correlation*. The matrix element $\langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle$ which represents the pairing interaction has been assumed, for simplicity, to be constant and its value G is the only additional parameter introduced in order to describe the pairing correlation. A dependence $\langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle$ on ν and ν' might be essential for a more detailed description. For example, the constancy of $\langle \nu\nu \,|\, G \,|\, \nu'\nu' \rangle$ (and, therefore, Δ_ν) leads in spherical nuclei to the same energy for all quasi-particles $E_\nu = \sqrt{\Delta_\nu^2 + (\varepsilon_\nu - \lambda)^2} = \frac{1}{2}\Omega G$ and, therefore, to a degeneracy of the excited states. A dependence of Δ_ν on ν eliminates this degeneracy. The residual interaction between quasi-particles causes the same effect and may also be important for a detailed analysis of single-particle spectra.

To simplify the problem, we did not distinguish between protons and neutrons. If we do not consider any pairing interaction between neutrons and protons, or if they occupy different shells, then the generalization of the problem is straightforward. The case with some neutron-proton pairing correlation included remains to be investigated.

For spherical nuclei, we have considered the idealized scheme of strongly degenerate levels removed from each other (shells). The validity of the present results for shells with a small number of states, as well as the effect of the splitting of a shell into subshells, will need further analysis.

Finally, it may be added that the pairing correlation may affect also other nuclear phenomena such as quadrupole and magnetic moments, electromagnetic transitions, etc.

* Only a few qualitative remarks have been made in I. 4 in order to indicate which parts of the nucleon-nucleon forces are responsible for the pairing interaction.

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Appendix A

Here, we write down the expression for $H_{\text{int}} = H_{40} + H_{31} + H_{22}$. For compactness, we rewrite (10) in the following way:

$$b_{\nu\sigma} = U_{\nu\sigma} x_{\nu\sigma} + V_{\nu\sigma} y^{+}_{\nu\sigma}, \qquad \text{(A. 1)}$$

where

$$\left. \begin{aligned} & x_{\nu+} = y_{\nu-} = \alpha_\nu; \qquad x_{\nu-} = y_{\nu+} = \beta_\nu \\ & U_{\nu\pm} = U_\nu; \qquad\qquad V_{\nu\pm} = \pm V_\nu . \end{aligned} \right\} \qquad \text{(A. 2)}$$

In this notation, H_{int} is given by

$$\left. \begin{aligned} H_{40} &= -\frac{1}{4} \sum \langle 12 | G | 2'\,1' \rangle U_1 U_2 V_{2'} V_{1'}\, x_1 x_2^+ y_{2'}^+ y_{1'}^+ + \text{conj.} \\ H_{31} &= -\frac{1}{2} \sum \langle 12 | G | 2'\,1' \rangle (U_1 U_2 V_{2'} U_{1'} - V_1 V_2 U_{2'} V_{1'})\, x_1^+ x_2^+ y_{2'}^+ x_{1'} \\ &\qquad + \text{conj.} \\ H_{22} &= -\frac{1}{4} \sum \langle 12 | G | 2'1' \rangle (U_1 U_2 U_{2'} U_{1'} + V_1 V_2 V_{2'} V_{1'})\, x_1^+ x_2^+ x_{2'} x_{1'} \\ &\quad + \sum \langle 12 | G | 2'1' \rangle U_1 V_2 V_{2'} U_{1'}\, x_1^+ y_{2'}^+ y_2 x_{1'}, \end{aligned} \right\} \qquad \text{(A. 3)}$$

where the indices correspond to ν and σ (e.g., $1 \equiv \nu_1 \sigma_1$) and the matrix elements are antisymmetrized

$$\langle 12 | G | 2'\,1' \rangle = \langle \nu_1 \sigma_1 \nu_2 \sigma_2 | G | \nu_2' \sigma_2' \nu_1' \sigma_1' \rangle - \langle \nu_1 \sigma_1 \nu_2 \sigma_2 | G | \nu_1' \sigma_1' \nu_2' \sigma_2' \rangle . \qquad \text{(A. 4)}$$

The matrix elements have the symmetry properties following from the definition of the conjugate states:

$$\langle \nu_1 \sigma_1 \nu_2 \sigma_2 \,|\, G \,|\, \nu_2' \sigma_2' \nu_1' \sigma_1' \rangle = \langle \nu_1 - \sigma_1 \nu_2 - \sigma_2 \,|\, G \,|\, \nu_2' - \sigma_2' \nu_1' - \sigma_1' \rangle^* \qquad (\text{A. }5)$$

from which it follows, in particular, that

$$\langle \nu_1 + \nu_2 - |\, G \,|\, \nu_2' - \nu_1' + \rangle = \langle \nu_2' + \nu_1' - |\, G \,|\, \nu_1 - \nu_2 + \rangle . \qquad (\text{A. }6)$$

Appendix B

Here, we shall calculate the nuclear energy (104) for small deformations of the self-consistent field. The main point will be to show that the effect of a variation of the single-particle wave functions with deformation may be neglected for the outside nucleons. The following calculations will explicitly exhibit also the procedure of the extraction of an additional self-consistent field and the choice of the new single-particle eigenstates ν, which demonstrate the nature of the first canonical transformation (3).

Let us look for the ground state of the Hamiltonian (101). The single-particle eigenstates ν in a deformed field are determined by the requirement of diagonalization of the single-particle energies $\tilde{\varepsilon}_{\nu\nu'}$, i.e., according to (94), by

$$\varepsilon_{\nu\nu'} - \tilde{\mu}\, q_{\nu\nu'} = \sum_k \varepsilon_k^0 \varphi_{k\nu}^* \varphi_{k\nu'} - \tilde{\mu} \sum_{kk'} q_{kk'}^0 \varphi_{k\nu}^* \varphi_{k'\nu'} = \tilde{\varepsilon}_\nu \delta_{\nu\nu'}$$

which can also be rewritten as

$$(\varepsilon_k^0 - \tilde{\varepsilon}_\nu)\, \varphi_{k\nu} = \tilde{\mu} \sum_{k'} q_{kk'}^0 \varphi_{k'\nu}, \qquad (\text{B. }1)$$

where the states k and the quantities ε_k^0 and $q_{kk'}^0$ correspond to the spherical field. We assume that the states k, corresponding to the same degenerate level, are chosen to make the matrix elements $q_{kk'}^0$ diagonal inside each shell. Assuming that $\varphi_{k\nu} = \delta_{k\nu} + \varphi_{k\nu}^{(1)}$ where the deviations from the spherical symmetry $\varphi_{k\nu}^{(1)}$ are small, we find in the first approximation

$$\varphi_{k\nu}^{(1)} = \frac{\tilde{\mu}\, q_{k\nu}^0}{\varepsilon_k^0 - \varepsilon_\nu^0} \qquad (\text{B. }2)$$

and, therefore,

$$\varepsilon_{\nu\nu} = \varepsilon_\nu^0 + \tilde{\mu}^2 \sum_{\nu'} \frac{|\, q_{\nu\nu'}^0 \,|^2}{\varepsilon_{\nu'}^0 - \varepsilon_\nu^0} \equiv \varepsilon_\nu^0 + \tilde{\mu}^2 p_\nu^0 \qquad (\text{B. }3)$$

$$q_{\nu\nu} = q^0_{\nu\nu} + 2\tilde{\mu}\sum_{\nu'}\frac{|q^0_{\nu\nu'}|^2}{\varepsilon^0_{\nu'} - \varepsilon^0_\nu} = q^0_{\nu\nu} + 2\tilde{\mu}\, p^0_\nu \qquad \text{(B. 4)}$$

$$\tilde{\varepsilon}_{\nu\nu} = \varepsilon_{\nu\nu} - \tilde{\mu}\, q_{\nu\nu} = \varepsilon^0_\nu - \tilde{\mu}\, q^0_{\nu\nu} - \tilde{\mu}^2\, p^0_\nu. \qquad \text{(B. 5)}$$

(For the closed shells, $\tilde{\mu} = \mu + \varkappa Q$ must be replaced by $\tilde{\mu}' = \mu' + \varkappa Q$). From (B. 5) it follows that the total splitting of the unfilled shell is given by (cf. (99))

$$\varepsilon'' - \varepsilon' = \tilde{\mu}\, q_0 + \tilde{\mu}^2\, p_0, \qquad \text{(B. 6)}$$

where

$$p_0 = p^0_\nu(\varepsilon_\nu = \varepsilon') - p^0_\nu(\varepsilon_\nu = \varepsilon''). \qquad \text{(B. 7)}$$

Using (B. 4), one obtains for the quadrupole moments

$$Q_{cl} = 2\tilde{\mu}'\sum_\nu{}' \, 2\, p^0_\nu \equiv 2\,\tilde{\mu}'\, P', \quad \Big(\sum_\nu{}' \, q^0_{\nu\nu} = 0\Big) \qquad \text{(B. 8)}$$

$$Q_\lambda = \sum_\nu{}^\lambda \, q^0_{\nu\nu}\, 2\, V^2_\nu + 2\,\tilde{\mu}\sum_\nu{}^\lambda \, p^0_\nu\, 2\, V^2_\nu. \qquad \text{(B. 9)}$$

The quantity V^2_ν depends on the total splitting of the λ-shell, i.e., on the parameter η. Therefore, we may write

$$\sum_\nu{}^\lambda \, q^0_{\nu\nu}\, 2\, V^2_\nu = q_0\, \Lambda(\eta) \qquad \text{(B. 10)}$$

$$\sum_\nu{}^\lambda \, p^0_\nu 2\, V^2_\nu = p_0\, \Lambda'(\eta). \qquad \text{(B. 11)}$$

The function $\Lambda(\eta)$ has been calculated earlier and is given by (100). The function $\Lambda'(\eta)$ is of the same order as $\Lambda(\eta)$.

The energy of non-interacting particles (the first term in (96)), is, according to (B. 3), given by

$$\sum_\nu \varepsilon_{\nu\nu}\, 2\, V^2_\nu = \sum_\nu \varepsilon^0_\nu\, 2\, V^2_\nu + \tilde{\mu}'^2\sum_\nu{}' \, 2\, p^0_\nu + \tilde{\mu}^2\sum_\nu{}^\lambda \, p^0_\nu\, 2\, V^2_\nu. \qquad \text{(B. 12)}$$

Note that the first term in the right hand side is constant, since

$$\sum_\nu \varepsilon^0_\nu\, 2\, V^2_\nu = \sum_\nu{}' \, 2\, \varepsilon^0_\nu + \varepsilon_\lambda\, N_\lambda \equiv W_0.$$

Using (B. 8) and (B. 11), we obtain

$$\sum_\nu \varepsilon_{\nu\nu}\, 2\, V^2_\nu = W_0 + \frac{1}{4\,P'}\, Q^2_{cl} + \tilde{\mu}^2\, p_0\, \Lambda'(\eta). \qquad \text{(B. 13)}$$

To estimate the expressions obtained above, we asume now that the states k correspond to the oscillator potential. In this model, the quantities q_0 and p_0 are given, in the usual notation, by

$$q_0 = \frac{3\,\hbar n}{m\omega}, \qquad p_0 = \frac{3\,\hbar n}{2\,m^2\,\omega^3}. \tag{B. 14}$$

(n is the principal quantum number). From (B. 14) and (B. 6) it follows that

$$\frac{\tilde{\mu}\,p_0}{q_0} = \tilde{\mu}\,q_0 \frac{p_0}{q_0^2} = \frac{\tilde{\mu}\,q_0}{6\,\hbar\omega n} \approx \frac{\varepsilon'' - \varepsilon'}{6\,\varepsilon_F} \ll 1, \tag{B. 15}$$

where ε_F is the Fermi energy. Within the accuracy of the small factor (B. 15), we may neglect the last terms in (B. 6) and (B. 9); then, these equations coincide with (99) and (98). The last term in (B. 13), which depends on η, is to be compared with other η-dependent terms in (96), say, $-\frac{1}{2}\varkappa Q_\lambda^2$. Then, we find

$$\frac{\tilde{\mu}\,p_0 \Lambda'}{\varkappa Q_\lambda^2} = \frac{\tilde{\mu}\,p_0}{q_0} \cdot \frac{\tilde{\mu}\Lambda'}{\varkappa Q_\lambda \Lambda}. \tag{B. 16}$$

Since $\Lambda' \sim \Lambda$ and $\tilde{\mu} \sim \varkappa Q_\lambda$, the ratio (B. 16) is of the order of the small factor (B. 15). Therefore, the last term in (B. 13) may be neglected, and this justifies the equation (103).

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